$\text{profit}(Q) = pQ - TC(Q)$

$\text{profit}: \mathbb{R}^1 \to \mathbb{R}^1$

Did the maximization.

$\text{profit}(K,L) = p \cdot f(K,L) - rK - wL$

$\text{profit}: \mathbb{R}^2 \to \mathbb{R}^1$

More generally, $f: \mathbb{R}^n \to \mathbb{R}^1, n \geq 1$

"Real valued functions"

Did maximization.

$y = f(x_1, \ldots, x_n)$

Consider a point $(x_1^0, \ldots, x_n^0)$

It is <u>differentiable</u> at this point if there is a unique tangent plane there.

If so, there are n "partial derivatives" at that point.

$$\frac{\partial f}{\partial x_1}(x_1^0, \ldots, x_n^0) = \lim_{h \to 0} \frac{f(x_1^0 + h, x_2^0, \ldots, x_n^0) - f(x_1^0, \ldots, x_n^0)}{h}$$

Get n of these, one for each direction parallel to each of n axes.

So, suppose $f(K,L) = 4 \cdot K^{3/4} \cdot L^{1/4}$

$$\frac{\partial f}{\partial K} = \left(\frac{3}{\cancel{4}}\right) \cdot \cancel{4} \cdot K^{\frac{3}{4}-1} \cdot L^{\frac{1}{4}}$$

$$= 3\left(L/K\right)^{1/4}$$

"Marginal product of capital"

$$\frac{\partial f}{\partial L} = \left(K/L\right)^{3/4}$$

"Marginal product of labor"

Implicit functions

$$y = -\frac{x}{2} + 2$$ "Explicit function"

$$x + 2y - 4 = 0$$ The same thing expressed "implicitly"

This implicit relationship between x and y is obviously a function.

$$\boxed{f(x,y) = 0}$$ Consider this in general

Is this implicit relationship a function?

$$x^2 + y^2 = -10$$

Written as $x^2 + y^2 + 10 = 0$ No solution

$$x^2 + y^2 = 10$$

Written as $x^2 + y^2 - 10 = 0$ Has solutions, not a function

A function may exist, but may not be differentiable

Our interest: Suppose a differentiable implicit function exists. What is its derivative?

That comes from the chain rule; surprisingly useful result.

If a differentiable implicit function exists, then this means that "for all x" there is a $y(x)$ such that

$$f[x, y(x)] = 0.$$

Sometimes write $f[x, y(x)] \equiv 0$ "identically zero," meaning over some range of values for x.

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial x} = 0, \Rightarrow \boxed{\frac{\partial y}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial y}}$$

Implicit functions

$$y = -\frac{x}{2} + 2 \Rightarrow \boxed{\frac{\partial y}{\partial x} = -\frac{1}{2}}$$

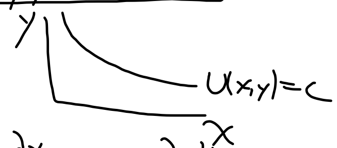
$$x + 2y - 4 = 0 \quad : \boxed{f(x,y) = x + 2y - 4}$$

$$f[x, y(x)] = x + 2\cdot\left[-\frac{x}{2} + 2\right] - 4 = 0, \text{ for all } x.$$

$$\boxed{\frac{\partial y}{\partial x} = \frac{-\partial f/\partial x}{\partial f/\partial y} = -\frac{1}{2}}$$

Consider marginal rates of substitution.

$$\boxed{U(x,y) - c = 0} \Rightarrow y(x), \frac{dy}{dx}$$



$$MRS_{xy} = \frac{\partial y}{\partial x} = -\frac{\partial U/\partial x}{\partial U/\partial y}$$

$$U(x,y) = x^{\beta} y^{1-\beta}, \quad 0 < \beta < 1$$

$$\boxed{MRS_{xy}\Big|_{y(x)} = -\frac{\beta}{1-\beta}\cdot\frac{y}{x}}\Bigg|_{y(x)} < 0$$

Solve this explicitly,

$$y = \left(\frac{c}{x^{\beta}}\right)^{\frac{1}{1-\beta}}$$

Verify for yourselves that these give you the same thing.

Constrained optimization

$$\min_{K,L} \; vK + wL$$
$$\text{s.t. } f(K,L) = \bar{Q}$$

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$$\max_{x,y} \; U(x,y)$$
$$\text{s.t. } P_x \cdot x + P_y \cdot y = I$$

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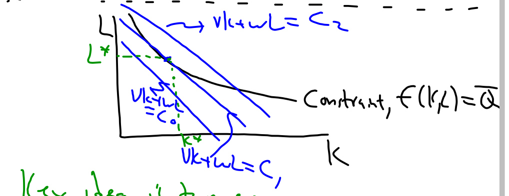

Key idea is tangency
Find slopes of the slice of the objective function and of the constraint, set equal to each other.

$$\max_{x_1,x_2} \; F(x_1, x_2)$$
$$\text{s.t. } G(x_1, x_2) = 0 \quad x_2(x_1)$$

implicit function.

Write this as an unconstrained maximization problem:

$$\max_{x_1} \; F\left[x_1, x_2(x_1)\right]$$

$$\frac{\partial F}{\partial x_1} + \frac{\partial F}{\partial x_2} \cdot \frac{\partial x_2}{\partial x_1} = 0$$

$$\Rightarrow \frac{\partial F}{\partial x_1} = -\frac{\partial F}{\partial x_2} \cdot \boxed{\frac{\partial x_2}{\partial x_1}}$$

$$\Rightarrow \frac{\partial F}{\partial x_1} = -\frac{\partial F}{\partial x_2} \cdot \left[-\frac{\partial G/\partial x_1}{\partial G/\partial x_2}\right]$$

$$\Rightarrow \boxed{\frac{\partial F/\partial x_1}{\partial F/\partial x_2} = \frac{\partial G/\partial x_1}{\partial G/\partial x_2}} \quad \text{"Tangency"}$$

1) $\dfrac{\partial F/\partial x_1}{\partial F/\partial x_2} = \dfrac{\partial G/\partial x_1}{\partial G/\partial x_2}$

2) $G(x_1, x_2) = 0$

$\Rightarrow$ 2 equations, 2 unknowns.
(x_1^*, x_2^*), solution to the constrained optimization problem.

One generalization:

$$\max_{x_1,\dots,x_n} F(x_1,\dots,x_n)$$

$$\text{s.t. } G(x_1,\dots,x_n) = 0$$

Introduces nothing new.

Method of Lagrange multipliers.

$$\mathcal{L}(x_1,\dots,x_n,\lambda) \equiv F(x_1,\dots,x_n) + \lambda[G(x_1,\dots,x_n)]$$

Then, find a stationary pt. of this.

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial F}{\partial x_1} + \lambda \frac{\partial G}{\partial x_1} = 0$$

$$\vdots$$

$$\frac{\partial \mathcal{L}}{\partial x_n} = \frac{\partial F}{\partial x_n} + \lambda \frac{\partial G}{\partial x_n} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = G(x_1,\dots,x_n) = 0$$

n+1 vbls.
n+1 eqns.

With $n=2$, $\dfrac{\partial F}{\partial x_1} + \dfrac{\lambda \partial G}{\partial x_1} = 0$

$\dfrac{\partial F}{\partial x_2} + \lambda \cdot \dfrac{\partial G}{\partial x_2} = 0$

$$\Rightarrow \frac{\frac{\partial F}{\partial x_1}}{\frac{\partial F}{\partial x_2}} = \frac{\cancel{-\lambda}\, \partial G/\partial x_1}{\cancel{-\lambda}\, \partial G/\partial x_2}$$

$$\max_{x_1, \ldots, x_n} \quad F(x_1, \ldots, x_n)$$

$$\text{s.t.} \quad G(x_1, \ldots, x_n) = 0$$

$$\underline{\text{and}} \quad H(x_1, \ldots, x_n) = 0$$

$$\Rightarrow \mathcal{L} = F(x_1, \ldots, x_n) + \lambda \left[G(x_1, \ldots, x_n) \right] + \gamma \left[H(x_1, \ldots, x_n) \right]$$

$$\left[\text{so, } \mathcal{L}\left(x_1, \ldots, x_n, \lambda, \gamma\right) \right]$$

$$\max_{x,y} \; U(x,y)$$

$$\text{s.t. } p_x x + p_y y = I$$

write it as $G(x,y)=0$

$$\mathcal{L}(x,y,\lambda) = U(x,y) + \lambda[p_x x + p_y y - I]$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial U}{\partial x} + \lambda p_x = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial U}{\partial y} + \lambda p_y = 0$$

$$\frac{\partial U/\partial x}{\partial U/\partial y} = \frac{p_x}{p_y}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = p_x x + p_y y - I = 0$$

$$-\frac{\partial U/\partial x}{\partial U/\partial y} = MRS_{xy}$$

Amt. of y you must give up if given 1 extra x, to hold utility fixed.

$$p_y y = I - p_x x$$

$$\Rightarrow y = \frac{I}{p_y} - \frac{p_x}{p_y} \cdot x$$

Our 3 equations, 3 unknowns give us an x^*, y^*, and λ^*.

Plug (x^*, y^*) back into utility fn.

$$x^*_E = x^*_E(p_x, p_y, I)$$

$$y^* = y^*(p_x, p_y, I)$$

$$U[x^*_E(p_x, p_y, I),\; y^*(p_x, p_y, I)]$$

$$\frac{\partial U^*}{\partial I} = \underbrace{\frac{\partial U^*}{\partial x}}_{\lambda p_x} \frac{\partial x^*}{\partial I} + \underbrace{\frac{\partial U^*}{\partial y}}_{\lambda p_y} \frac{\partial y^*}{\partial I}$$

$$\Rightarrow \boxed{\lambda = \frac{\partial U^*}{\partial I}}$$