Discontinuous Payoffs, Shared Resources, and Games of Fiscal Competition: Existence of Pure Strategy Nash Equilibrium

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Abstract

We define a class of games with discontinuous payoffs that we call shared resource games and establish a pure strategy Nash equilibrium existence theorem for these games. We then apply this result to a canonical game of fiscal competition for mobile capital. Other applications are also discussed. Our result for the mobile capital game holds for any finite number of regions, permits general preferences over private and public goods, and does not assume that production technologies have a particular functional form, or are identical in all regions, or satisfy the Inada condition at zero.

1. Introduction

We define a class of games with discontinuous payoffs and develop a general theorem on the existence of pure strategy Nash equilibrium in these games. We then apply this result to a canonical game of fiscal competition for mobile capital, specifically to a generalization of the game in Laussel and Le Breton (1998). Related work and ongoing research show that the class of applications is much wider (Rothstein 2005).

Anyone familiar with the literature on fiscal competition will immediately wonder why a result on discontinuous games would be helpful for establishing existence. The games in the large applied literature have continuous payoffs

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(e.g., Zodrow and Mieszkowski 1986, Wilson 1986, Wildasin 1988, 1991). So do the games in the small theoretical literature on existence (Bucovetsky 1991, Laussel and Le Breton 1998, Dhillon, Wooders, and Zissimos 2004). The difference occurs because our model varies two assumptions that are used in the theoretical literature. First, we use ad valorem taxes instead of unit taxes. The former are more convenient in the analysis of existence since the strategy spaces have natural bounds. Lockwood (2004) provides a recent discussion of other differences. Second, we assume that the aggregate amount of mobile capital is fixed instead of variable. This is the standard assumption in the applied literature. Together, these changes produce at least one, and possibly many, discontinuity points that we argue are fundamental. On the other hand, the analysis of quasiconcavity becomes much more intuitive and straightforward.

Losing continuity to gain an easier analysis of quasiconcavity proves to be a worthwhile tradeoff. Our existence result holds for any finite number of regions, permits general preferences over private and public goods, and does not assume that production technologies have a particular functional form, or are identical in all regions, or satisfy the Inada condition at zero. Recent work on fiscal competition emphasizes the substantive importance of allowing for regional heterogeneity and asymmetric equilibria (see Cai and Treisman 2005, and the references therein). The production functions do need to satisfy a number of restrictions. Many of these conditions have an economic interpretation, and we show that certain familiar technologies satisfy them.

We call the class of games to which our general theorem applies *shared resource games*. In these games, players compete for a share of a resource that is in fixed total supply, except perhaps at certain joint strategies. Each player's payoff depends on other players' strategies only through the effect those strategies have on the amount of the shared resource that the player obtains. That is to say, each player's payoff function u_i has the form $u_i(x_i, x_{-i}) = F_i[x_i, S_i(x_i, x_{-i})]$. The continuity and quasiconcavity properties of u_i are derived from the relevant properties of F_i and S_i.

While we could tailor an existence proof to the mobile capital game, it makes sense to establish a more general theorem. The mobile capital game is very simple. Each region has just one tax base, so the strategy spaces are all one dimensional. Other games of fiscal competition allow each region to use many tax bases. Our general theorem is useful for studying existence in this case (multiple tax bases are examined in Rothstein 2005). Furthermore, the only discontinuity point in the mobile capital game occurs when all tax rates are too high. This property results directly from the fact that the owners of the mobile factor (capital owners) receive no benefit from the local public good. This is a special assumption. If the local public good were infrastructure or the mobile factor were labor, then owners of the mobile factor would value the local public good. Discontinuities could then occur when tax rates in all regions were too low, too high, or some combination of the two. Our general theorem is useful for studying existence in these models as well (mobile labor is examined in Rothstein 2005). There are also games like Tullock's rent seeking game, which we discuss below, that are shared resource games but do not fit neatly into the fiscal competition framework.

Our existence theorem for shared resource games requires the most general result in Reny (1999). We explain in detail why earlier results are not powerful enough. Furthermore, our result is easier to apply in some cases. To use Reny's result, one must analyze the closure of the graph of the vector payoff function of the game. When there are just two players each with one-dimensional strategy spaces, the graph itself consists of points of the form $(x_1, x_2, u_1(x_1, x_2), u_2(x_1, x_2))$. This is a relatively high-dimensional object. Then, one must find its closure. Our result requires no analysis of any closures of high-dimensional objects. This is not to say that its requirements are trivial, but they can be easier to establish.

In the next section, we define shared resource games and discuss the discontinuities that are allowed. Section 3 analyzes quasiconcavity in these games, and Section 4 gives the main existence result. Section 5 gives a fully formal development of a canonical game of fiscal competition for mobile capital and derives conditions that ensure the existence of a pure strategy Nash equilibrium. Section 6 draws conclusions about the structure of games of fiscal competition and considers directions for further research.

2. Shared Resource Games

2.1. Model

There are $N \geq 2$ players each with *pure strategy space* X_i, $i = 1, 2, \ldots, N$. Let $\mathcal{X} \equiv \times_{i=1}^{N} X_i$ be the *joint strategy space.*

ASSUMPTION 1 (strategy spaces): *For all i, X_i contains more than one point and is a compact and convex subset of $\Re^k$, $k \geq 1$.*[1]

Associated with each agent is a function $S_i : \mathcal{X} \to [0, \bar{s}]$, where $0 < \bar{s} < \infty$. This is the *sharing rule for i.* Define $\mathcal{D}_i \subset \mathcal{X}$ to be the (possibly empty) set of joint strategies at which S_i is discontinuous. The set $\Delta = \bigcup_{i=1}^{N} \mathcal{D}_i$ is then all of the joint strategies at which one or more of the sharing rules is discontinuous. We call Δ the set of *degenerate joint strategies.* The set $\mathcal{X} \backslash \Delta$ is all of the joint strategies at which all of the sharing rules are continuous. We call $\mathcal{X} \backslash \Delta$ the set of *nondegenerate joint strategies.*

Assumption 2 places further restrictions on these sets and the sharing rules.

[1]Note that the dimension of X_i can vary across agents, so the interior of X_i in $\Re^k$ may be empty. Under Assumption 1, however, the relative interior and relative boundary must be nonempty. One needs this degree of generality to analyze games of fiscal competition in which different regions have different numbers of tax bases (Rothstein 2005).

ASSUMPTION 2:

i. *For all* $x \in \mathcal{X} \setminus \Delta$, $\sum_{i=1}^{N} S_i(x) = \bar{s}$.

ii. *There exists* $\underline{s} \in [0, \bar{s}]$ *such that for all* $x \in \Delta$, $\sum_{i=1}^{N} S_i(x) = \underline{s}$.

iii. *For all* i, *all* $(x_i, x_{-i}) \in \mathcal{D}_i$ *and every neighborhood of* x_i, *there exists* $x_i' \in X_i$ *such that* $(x_i', x_{-i}) \in \mathcal{X} \setminus \mathcal{D}_i$.

iv. *For all* i, *there exists a constant* $\tilde{s}_i$ *satisfying* $\bar{s} \geq \tilde{s}_i > \frac{\bar{s}}{N}$ *such that for all* $(x_i, x_{-i}) \in \Delta$ *and all* $(x_i', x_{-i}) \in \mathcal{X} \setminus \mathcal{D}_i$, $S_i(x_i', x_{-i}) \geq \tilde{s}_i \geq S_i(x_i, x_{-i})$.

Assumptions $2(i)$ and $2(ii)$ fix the aggregate amount of the shared resource at nondegenerate and degenerate joint strategies, respectively. Note that Assumption $2(ii)$ introduces the parameter $\underline{s}$. This allows us to examine how technical changes in a game at degenerate joint strategies do (or do not) affect the equilibria of the game.

Assumption $2(iii)$ states that each player can make a small unilateral deviation from any of her discontinuity points and arrive at one of her continuity points.

Assumption $2(iv)$ says that there exists a player-specific value, $\tilde{s}_i$, that weakly separates the shares the player obtains at all degenerate joint strategies from the shares she obtains from all unilateral deviations from such a point (more precisely, from unilateral deviations to any of her continuity points). This value must exceed an equal share of the shared resource. The latter requirement is weak if N is large.

Example 1 illustrates these ideas. We use it a number of times below.

Example 1: $N = 2$, $X_i = [0, 1]$, $\bar{s} = 1$, and

$$S_1(x_1, x_2) = \begin{cases} \dfrac{x_1^2(1-x_1)^2}{x_1^2(1-x_1)^2 + x_2^2(1-x_2)^2} & \text{otherwise} \\ 0.25, & x_1 = 0 \text{ and } x_2 = 0 \\ 0.25, & x_1 = 0 \text{ and } x_2 = 1 \\ 0.75, & x_1 = 1 \text{ and } x_2 = 0 \\ 0.75, & x_1 = 1 \text{ and } x_2 = 1 \end{cases}$$

$$S_2(x_1, x_2) = \begin{cases} \dfrac{x_2^2(1-x_2)^2}{x_1^2(1-x_1)^2 + x_2^2(1-x_2)^2} & \text{otherwise} \\ 0.75, & x_1 = 0 \text{ and } x_2 = 0 \\ 0.75, & x_1 = 0 \text{ and } x_2 = 1 \\ 0.25, & x_1 = 1 \text{ and } x_2 = 0 \\ 0.25, & x_1 = 1 \text{ and } x_2 = 1 \end{cases}$$

The two sharing rules are clearly discontinuous at the corners of the joint strategy space, so $\mathcal{D}_1 = \mathcal{D}_2 = \Delta = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$.

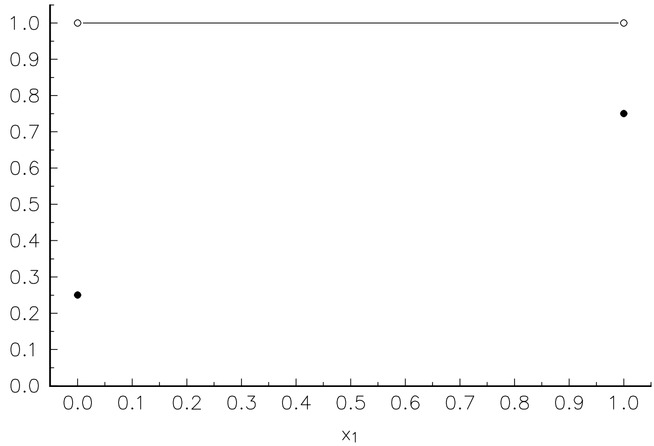

Figure 1: $S_1(x_1, 0)$ and $S_1(x_1, 1)$

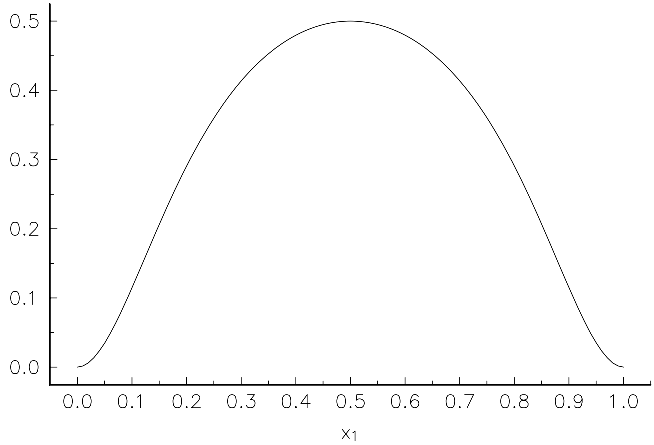

Figure 2: $S_1(x_1, 0.5)$

Assumptions $2(i)$–$2(ii)$ are immediate with $\underline{s} = 1$. Assumption $2(iii)$ holds since for every agent i and $(x_i, x_{-i}) \in \mathcal{D}_i$, there is always a point x_i' in $[0, 1]$ (any point in $(0, 1)$ works) such that S_i is continuous at (x_i', x_{-i}). To establish Assumption $2(iv)$, define $\tilde{s}_i = 1$ for both players. We obviously have $\tilde{s}_i > 1/2 = \bar{s}/N$. For each i and $(x_i, x_{-i}) \in \Delta$, we have $\tilde{s}_i > 0.75 \geq S_i(x_i, x_{-i})$. A deviation to any $x_i' \in (0, 1)$ gives $S_i(x_i', x_{-i}) = 1 = \tilde{s}_i$. The required inequalities therefore hold.

Each player has a payoff function $u_i : \mathcal{X} \to \Re$. We now impose the assumption that player i is affected by the strategies of the *other* players entirely through the sharing rule.

ASSUMPTION 3 (representation of payoff functions): *For each player i, there exists a function $F_i : X_i \times [0, \bar{s}] \to \Re$ such that for all $x \in \mathcal{X}$,*

$$u_i(x) = F_i[x_i, S_i(x)].$$

ASSUMPTION 4 (properties of F_i): *For all i,*

i. *F_i is continuous.*

ii. *For all $x_i \in X_i$, $F_i(x_i, \cdot)$ is nondecreasing in s_i.*

iii. *Given any $s_i' > \frac{\bar{s}}{N}$, $\max_{x_i \in X_i} F_i(x_i, s_i') > \max_{x_i \in X_i} F_i(x_i, \frac{\bar{s}}{N})$.*

Note that Assumption 4(*iii*) would follow from the simpler but much stronger requirement that a player can achieve strictly higher utility with strictly more of the shared resource.

Example 2: Suppose $F_i = x_i s_i$, where $X_i = [0, 1]$. Given any $s_i' > \frac{\bar{s}}{N}$, we have $\max_{x_i \in X_i} F_i(x_i, s_i') = F_i(1, s_i') = s_i' > \bar{s}/N = F_i(1, \bar{s}/N) = \max_{x_i \in X_i} F_i(x_i, \frac{\bar{s}}{N})$.

We have not yet imposed conditions assuring that $u_i(\cdot, x_{-i})$ is quasiconcave on X_i. No single sufficient condition holds in all of our applications. We therefore build quasiconcavity into the definition of shared resource games and return to the issue further on.

ASSUMPTION 5: *For all i and all x_{-i}, $u_i(\cdot, x_{-i})$ is quasiconcave on X_i.*

DEFINITION. *A game $G = (X_i, u_i)_{i=1}^N$ with strategy spaces and payoffs satisfying Assumptions 1–5 is a shared resource game.*

2.2. Discontinuities in Shared Resource Games

What kinds of payoff function discontinuities can occur in shared resource games? We show in this section that discontinuities can occur that most existence theorems forbid. Payoff functions need not be upper semicontinuous. This rules out using the results in Dasgupta–Maskin (1986). Best replies need not exist for certain players at certain points. This rules out using Milgrom and Roberts (1990). Finally, the games need not be "reciprocally upper semicontinuous," which roughly says that some or all payoff discontinuities must be upward jumps.[2] This rules out using Simon (1987) and all but the most powerful result in Reny (1999).[3]

[2]A game is *reciprocally upper semicontinuous* if, whenever (x^*, u^*) is in the closure of the graph of its vector payoff function and $u_i(x^*) \leq u_i^*$ for every player i, then $u_i(x^*) = u_i^*$ for every player i . That is to say, having only downward jumps ($u_i(x^*) < u_i^*$ for some agents and $u_i(x^*) = u_i^*$ for all other agents) is not allowed, but having only upward jumps is.

[3]In principle, the theorems developed in Baye et al. (1993) are applicable. They proved difficult to apply in our framework. The work of Simon and Zame (1990) is directly applicable, but only to establish the existence of a Nash equilibrium in mixed strategies with an endogenous sharing rule. Reny (1999) also provides a result for mixed strategy equilibrium.

We establish these claims with a simple but abstract example. Rothstein (2005) presents a game of fiscal competition with the same properties.

Example 3: Consider the game $G = (X_i, u_i)_{i=1}^2$ with strategy spaces and S_i from Example 1, $F_i = x_i s_i$ from Example 2, and $u_i(x_1, x_2) = F_i[x_i, S_i(x_1, x_2)]$. This is a shared resource game: we have already established Assumptions 1–4, and we leave the proof of quasiconcavity for Section 3 below.

Consider the joint strategy $(1, 0)$. Notice first that Player 1's sharing rule and payoff function fail to be upper semicontinuous in her own strategy. If $0 < x_1 < 1$ then $S_1(x_1, 0) = 1$, so the payoff function is $u_1(x_1, 0) = (x_1)(1) = x_1$. We then have $S_1(1, 0) = 0.75$, so $u_1(1, 0) = (1)(0.75) = 0.75$. Thus, Player 1's payoff approaches 1 from below and then jumps down.

It is immediately clear that Player 1 does not have a best reply against $x_2 = 0$. A best reply would exist if and only if $u_1(0, 0) \geq 1$, but in fact $u_1(0, 0) = 0$. Roughly speaking, Player 1's payoff function jumps down at precisely the point at which it needs to be continuous or jump up.

Finally, there is no upward jump in Player 2's payoff at $(1, 0)$. It is true that her *sharing rule* jumps up. If $0 < x_1 < 1$ then $S_2(x_1, 0) = 0$, but $S_2(1, 0) = 0.25$. Her *payoff function*, however, is continuous in x_1 there, since $\lim_{x_1 \to 1} u_2(x_1, 0) = \lim_{x_1 \to 1}(0) S_2(x_1, 0) = 0 = (0) S_2(1, 0) = u_2(1, 0)$. This is sufficient to show that the game is not reciprocal upper semicontinuous.[4] The point $(1, 0)$ therefore illustrates a kind of triple failure: a failure of upper semicontinuity, of existence of a best reply, and of reciprocal upper semicontinuity of the game.

Besides these examples, there is an additional reason for not restricting attention to games that satisfy reciprocal upper semicontinuity. Suppose we are given a shared resource game with payoff discontinuities that satisfies this condition. One can generally make merely technical variations in the sharing rules and create another shared resource game that fails it. Upward jumps in agents' payoffs can only result from upward jumps in resource shares at degenerate joint strategies. We can eliminate *all* of these upward jumps by reducing the amount of the shared resource that is allocated to these agents at these points. This is just a technical change when there is no compelling reason for a particular allocation, as seems likely. The new game would have a smaller value of $\underline{s}$, but again this is likely to be just a technical change. Without the upward jumps, the modified game must fail reciprocal upper semicontinuity. We want both the original and the modified game to lie within the scope of our analysis.

[4]Formally, consider the sequence of joint strategies $\{(x_1^j, x_2^j)\}_{j=1}^\infty = \{(1 - (1/2j), 0)\}_{j=1}^\infty$. This converges to $(x_1^*, x_2^*) = (1, 0)$. It generates a sequence of payoff vectors $(u_1(x_1^j, x_2^j), u_2(x_1^j, x_2^j))$ that, as argued in the text, converges to $(u_1^*, u_2^*) = (1, 0)$. The point $(x_1^*, x_2^*, u_1^*, u_2^*) = (1, 0, 1, 0)$ is therefore in the closure of the graph of the vector payoff function. We have $u_1(x_1^*, x_2^*) = 0.75 < 1 = u_1^*$, and $u_2(x_1^*, x_2^*) = 0 = u_2^*$. Given the first inequality, the game would be reciprocally upper semicontinuous only if $u_2(x_1^*, x_2^*) > u_2^*$.

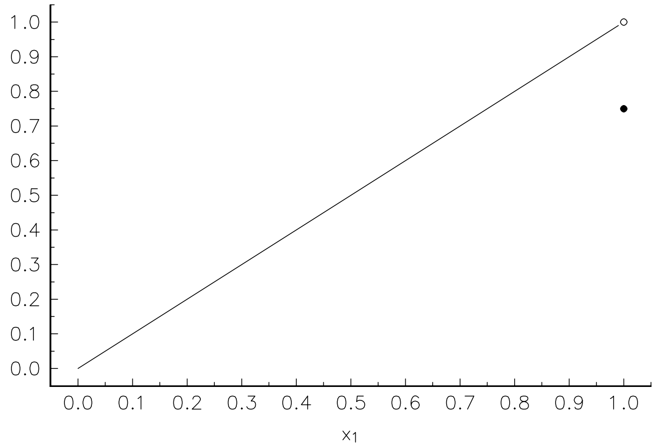

Figure 3: $u_1(x_1, 0)$

2.3. Discontinuities in Games of Fiscal Competition

Section 5 presents a fully formal proof of the existence of equilibrium in a standard game of fiscal competition for mobile capital. In that game, there is a single joint strategy of tax rates at which some payoff functions are not upper semicontinuous and best replies do not exist. Intuitively, suppose that all but one region levies a 100% tax on capital. Under the standard assumption that the marginal product of capital is always strictly positive, the region with the low tax rate has the entire capital stock. Its payoff function also strictly increases in its own tax rate, as in Figure 3. The payoff generally jumps down once the tax rate equals 100%.[5]

This single discontinuity in the mobile capital game is not a triviality. It has important implications and is not easily removed. It is sufficient to render useless the standard existence results, as explained in the previous section. One may be tempted to eliminate it by "trimming" the strategy space, but this approach is not satisfactory. The problem is that there is no obvious way to do this and also to ensure that the modified game has an *interior* equilibrium. This is a minimal requirement for the modified game to be an interesting replacement for the original game. Boundary equilibria on an arbitrary boundary are equally arbitrary.

There is another reason not to tailor a solution to the single discontinuity point in the mobile capital game. Variations on this game and other games of fiscal competition can have far more discontinuities. In the mobile capital game, capital owners derive no benefits from local public goods. This is a

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[5]The jump down would not occur if this region happens to be allocated the entire capital stock when all regions levy a 100% tax. In this case we could repeat the analysis for any other region and there would have to be a jump down in its payoff.

special assumption. If the local public good were infrastructure or the mobile factor were labor, then owners of the mobile factor would value the local public good. This could lead to discontinuities when tax rates in all regions were too low, too high, or some combination of both.

Rothstein (2005) develops a game of fiscal competition for mobile labor along these lines. It has a continuum of payoff discontinuities. It also has the same properties as the abstract game in Example 3: the payoff functions are not upper semicontinuous, a best reply does not exist to certain strategies, and the game is not reciprocally upper semicontinuous.

3. Quasiconcavity in Shared Resource Games

The following well-known theorem provides one starting point for the analysis of quasiconcavity in shared resource games. We conserve notation in stating it. Also, when we state that $S_i(\cdot, x_{-i})$ is concave or continuous, we mean regarding it as a function of x_i alone, and that the property holds on all of X_i.

THEOREM 1: *Suppose we are given Assumption 1, a function $S_i : \mathcal{X} \to [0, \bar{s}]$ with $S_i(\cdot, x_{-i})$ concave and $0 < \bar{s} < \infty$, and a function $F_i : X_i \times [0, \bar{s}] \to \Re$ with $F_i(x_i, s_i)$ quasiconcave in both arguments and nondecreasing in s_i. Then the composition $F_i[\cdot, S_i(\cdot, x_{-i})]$ is quasiconcave on X_i.*

Concavity of $S_i(\cdot, x_{-i})$ is a strong assumption. There is, however, at least one interesting shared resource game in which this condition holds.

Example 4 (Tullock's Rent Seeking Game; Baye et al. 1993): Assume any finite number of players $N \geq 2$ with strategy spaces $X_i = [0, 1]$ and payoffs:

$$u_i(x_i, x_{-i}) = \begin{cases} 1/N & x_j = 0, \ j = 1, \dots, N \\ \dfrac{x_i^\alpha}{\sum_{j=1}^{N} x_j^\alpha} - x_i & \text{otherwise} \end{cases}$$

with $1 > \alpha > 0$.[6]

Define $\bar{s} = 1$ and $S_i : [0, 1]^N \to [0, 1]$ with

$$S_i(x_i, x_{-i}) = \begin{cases} 1/N & x_j = 0, \ j = 1, \dots, N \\ \dfrac{x_i^\alpha}{\sum_{j=1}^{N} x_j^\alpha} & \text{otherwise.} \end{cases}$$

[6]The usual interpretation of $x_i^\alpha / \sum_{j=1}^{N} x_j^\alpha$ is that it represents the probability player i wins a prize worth 1 unit by spending x_i units. One may also interpret it as the share of the prize that player i obtains with certainty by spending x_i, since this gives the same payoff function.

Also define $F_i : [0, 1]^2 \to \Re$ with $F_i(x_i, s_i) = s_i - x_i$. With these definitions, the composition $F_i[x_i, S_i(x_i, x_{-i})]$ is equivalent to $u_i(x_i, x_{-i})$ from the Tullock game.

We now show that the Tullock game is a shared resource game. Assumptions 1–4 are straightforward, just note that, for all i, $\mathcal{D}_i = \Delta = \{(0, \ldots, 0)\}$, $\tilde{s}_i = 1$, and $\underline{s} = 1$. To apply Theorem 1, we need F_i quasiconcave. This is immediate from linearity. We also need $S_i(\cdot, x_{-i})$ concave on X_i for all x_{-i}. For $x_{-i} \neq 0$ this can be shown by taking derivatives. For $x_{-i} = 0$ it is obvious. It follows that $u_i(\cdot, x_{-i})$ is quasiconcave in all cases so this is a shared resource game.

To weaken the concavity requirement, we distinguish two cases. Suppose first that $S_i(\cdot, x_{-i})$ is continuous. We find in this case that also requiring $S_i(\cdot, x_{-i})$ to be concave is too strong. It would rule out the sharing rules in Example 1 (recall Figure 2) and those in Rothstein (2005). We weaken this requirement in Assumption 6. The basic idea is to transfer some of the curvature of F_i over to $S_i(\cdot, x_{-i})$ and show that the transformed functions are quasiconcave and concave, respectively.[7]

On the other hand, suppose $S_i(\cdot, x_{-i})$ is discontinuous at some point in X_i. In this case we still require $S_i(\cdot, x_{-i})$ to be concave. This immediately implies that discontinuities in own strategies must lie on the relative boundary of X_i (Rockafellar 1970, Theorem 23.1).

It may seem surprising that we require concavity in this case given its strong implications. Our justification is that the strategic variables in which we are mainly interested are tax rates. We only expect discontinuities when all tax rates are extreme in some sense (too low or too high). These rates naturally lie on the boundary of the individual strategy spaces. Note also that we are simply developing a sufficient condition for the quasiconcavity of $u_i(\cdot, x_{-i})$. Other approaches may exist with different implications for the location of the discontinuity points of $S_i(\cdot, x_{-i})$.

ASSUMPTION 6 (quasiconcavity): *For all i,*

i. F_i *is quasiconcave on* $\mathrm{ri}(X_i) \times (0, \bar{s})$.

ii. *Fix any* $x_{-i} \in \mathcal{X}_{-i}$ *such that* $S_i(\cdot, x_{-i})$ *is continuous. Then there exists a real-valued, increasing and continuous function* $T : (0, \bar{s}) \to \hat{\mathcal{S}} \subset \Re$, *written as* $T(s_i) = \hat{s}_i$ *(we suppress the dependence on* x_{-i}*), such that*

   (*a*) $T\,[S_i(\cdot, x_{-i})]$ *is concave on* $\mathrm{ri}(X_i)$.

   (*b*) $F_i[\cdot, T^{-1}(\cdot)]$ *is quasiconcave on* $\mathrm{ri}(X_i) \times \hat{\mathcal{S}}$.

iii. *Fix any* $x_{-i} \in \mathcal{X}_{-i}$ *such that* $S_i(\cdot, x_{-i})$ *is not continuous. Then* $S_i(\cdot, x_{-i})$ *is concave.*

[7]We developed this condition before discovering Fabella (1992), who generalizes the approach.

Note that Assumption 6(i) only requires F_i to be quasiconcave on the relative interior of its domain. This condition with Assumption 4(i) (continuity) implies quasiconcavity on the entire domain. The weaker condition is easier to check in applications when F_i is differentiable. For the same reason, Assumption 6(ii) only requires $T[S_i(\cdot, x_{-i})]$ to be concave on the relative interior of X_i.

We now have the following:

THEOREM 2: *A game $G = (X_i, u_i)_{i=1}^N$ satisfying Assumptions 1–4 and Assumption 6 is a shared resource game. For all i, any discontinuity points of $S_i(\cdot, x_{-i})$ must lie on the relative boundary of X_i.* [8]

We illustrate Assumption 6 by continuing Example 3.

Example 3 (continued): As noted above, $S_i(\cdot, x_{-i})$ is clearly not concave (recall Figure 2), so we cannot use Theorem 1. Fix $i = 1$ (the case $i = 2$ is the same). We have $F_1 = x_1 s_1$, so Assumption 6(i) is clear. If $x_2 \in (0, 1)$ then $S_1(\cdot, x_2)$ is continuous, so in these cases we need to establish Assumption 6(ii). Choose $T(s_1) = \ln(s_1)$. The composition $T[S_1(\cdot, x_2)]$ is well defined since $0 < S_1(\cdot, x_2) < \bar{s}$ on $(0, 1)$. It is straightforward to verify that this is concave on $(0, 1)$. For part (b), we need to verify that $F_1[x_1, T^{-1}(\hat{s}_1)] = x_1 \exp(\hat{s}_1)$ is quasiconcave on $(0, 1) \times \hat{\mathcal{S}}$, where $\hat{\mathcal{S}} = (-\infty, \ln(\bar{s}))$. Taking logs gives $\ln(x_1) + \hat{s}_1$. This is obviously concave in $(x_1, \hat{s}_1)$, so (b) follows. Now suppose $x_2 = 0$ or $x_2 = 1$. Then $S_1(\cdot, 0) = S_1(\cdot, 1) = 1$ on $(0, 1)$ but 0.25 or 0.75 at the endpoints. In all cases, $S_1(\cdot, x_2)$ is discontinuous and concave (recall Figure 1).

4. Existence of Pure Strategy Nash Equilibrium

The following definitions are in Reny (1999). Denote the vector of the players' payoff functions by $u : \mathcal{X} \to \Re^N$, where $u(x) = (u_1(x), \ldots, u_N(x))$ for every $x \in \mathcal{X}$. The graph of the vector payoff function is the subset of $\mathcal{X} \times R^N$ given by $\{(x, u) \in \mathcal{X} \times \Re^N | u = u(x)\}$.

DEFINITION: *Player i can* secure *a payoff $\alpha \in \Re$ at $x \in \mathcal{X}$ if there exists $\bar{x}_i \in X_i$, such that $u_i(\bar{x}_i, x'_{-i}) \geq \alpha$ for all x'_{-i} in some neighborhood of x_{-i}.*

DEFINITION: *A game $G = (X_i, u_i)_{i=1}^N$ is* better-reply secure *if whenever (x^*, u^*) is in the closure of the graph of its vector payoff function and x^* is not an equilibrium, some player i can secure a payoff strictly above u_i^* at x^*.*

[8] One can also show that $\Delta \subset \mathrm{rb}(\mathcal{X})$. The proof uses Theorem 2 and the fact that at any point in Δ, at least one sharing rule must be discontinuous in its own strategy.

DEFINITION: *A game* $G = (X_i, u_i)_{i=1}^N$ *is* quasiconcave *if each* X_i *is convex and for each* i *and every* $x_{-i} \in X_{-i}$, $u_i(\cdot, x_{-i})$ *is quasiconcave.*

DEFINITION: *A game* $G = (X_i, u_i)_{i=1}^N$ *is* compact *if each* X_i *is a nonempty compact subset of a topological vector space and each payoff function is bounded.*

THEOREM (Theorem 3.1, Reny 1999): *If* $G = (X_i, u_i)_{i=1}^N$ *is better-reply secure, quasiconcave and compact, then it possesses a pure strategy Nash equilibrium.*

Our first result is:

THEOREM 3: *Every game satisfying Assumptions 1–5 (i.e., every shared resource game) has at least one pure strategy Nash equilibrium.*

The proof rests on three ideas. First, given any degenerate joint strategy x^* and any sequence in $\mathcal{X}$ converging to it, there must be some player whose "limit payoff" (u_i^*) is no higher than a certain upper bound. This upper bound is the maximum utility this person could achieve if she were exogenously assigned an equal share of the shared resource. For example, in the two-player rent-seeking game, this upper bound is 0.5. A player given an equal share of the resource has utility function $F_i(x_i, 0.5) = 0.5 - x_i$. She does best by choosing $x_i = 0$ and achieves a maximum utility of 0.5. Some player's limit payoff must be no higher than this bound. If not, then we can show that at payoff vectors sufficiently close to u^*, everyone would have strictly more than an equal share of the resource. This is not possible.[9]

The second idea is that the player with the bounded limit payoff can deviate and realize a payoff above the bound. This involves showing that the player can deviate in a way that provides her with a large increase in the amount of shared resource at just a small cost. This makes her strictly better off. The result follows formally from the interplay of Assumptions 4(*iii*) and 2(*iv*).

Last, we construct the deviation so that the deviator's payoff function is continuous at the deviation point. She achieves a payoff strictly above her limit payoff at this point, and with continuity it follows that she can also secure strictly more than her limit payoff at this point. This is what we needed to show.

Theorem 4 shows that all sharing rules, and therefore all payoff functions, are continuous at equilibrium points. This means that the search for equilibria in shared resource games can be restricted to the set of nondegenerate joint strategies. This raises the question of whether we can also rule out from the search any individual strategy that happens to be part of a degenerate joint

[9]It may be possible to construct a sequence of payoffs for a given player that gives her a limit payoff above this bound, and by constructing different sequences one may be able to do this for all players. The issue, however, is whether there is a single sequence that does this for all players.

strategy. The Appendix contains a supplementary example after the proof of Theorem 4 that shows that the answer is no.

THEOREM 4: *If x^* is a pure strategy Nash equilibrium of a shared resource game, then $x^* \in \mathcal{X} \setminus \Delta$.*

5. Fiscal Competition for Mobile Capital

We now consider the existence of equilibrium in a standard game of fiscal competition for mobile capital (Wildasin 1988, 1991), modified for ad valorem taxation. There are $N \geq 2$ players, one in each region. The player in each region owns the fixed factors there. Fixed factors and mobile capital combine in region i to produce output. The production function in region i is $f_i : [0, a_i) \to \Re_+$, where $a_i > 0$ and "+" means "nonnegative." Fix the parameter $\bar{s}$, with $0 < \bar{s} < a_i$. This denotes the total amount of capital that is potentially available; the precise meaning of this is made clear below. $f_i(s_i)$ denotes a level of output.[10]

ASSUMPTION 7: *Assume for all i,*

i. $f_i(0) = 0$ *and* f_i *is continuous on* $[0, \bar{s}]$ *and* C^3 *on* $(0, \bar{s}]$.

ii. $\infty > f_i' > 0$ *on* $(0, \bar{s}]$.

iii. $-\infty < f_i'' < 0$ *on* $(0, \bar{s}]$.

The owner in region i pays for the use of capital, and these payments are then taxed at rate $t_i \in [0, 1] = X_i$ to fund local public goods. Let m_i denote gross payments to capital. Then $m_i(s_i) = s_i f_i'(s_i)$ for $s_i > 0$ and $m_i(0) = 0$. Private consumption good, c_i, equals output less the owner's payments to capital, so $c_i = f_i(s_i) - m_i(s_i)$. We assume the regional budget is balanced, so the amount of local public good, z_i, equals tax payments, giving $z_i = t_i m_i(s_i)$. The owner's preferences over private and public good are represented by the utility function $U_i : \Re_+^2 \to \Re$, written as $U_i(c_i, z_i)$.

In the game of fiscal competition for mobile capital, each player i unilaterally chooses the tax on capital in her region, $t_i \in [0, 1] = X_i$. Conditional on a quantity of capital in region i, preferences over tax rates may be derived by substituting the previous expressions into the utility function.[11] This gives $F_i : X_i \times [0, \bar{s}] \to \Re$, where

$$F_i(t_i, s_i) = U_i[f_i(s_i) - m_i(s_i),\ t_i m_i(s_i)].$$

[10]The domain of f_i is $[0, a_i)$ instead of $[0, \bar{s}]$ so the derivative of f_i at $\bar{s}$ denotes a standard two-sided derivative and not a right derivative. Derivatives are by definition finite but right and left derivatives may be infinite. This terminology follows Rockafellar (1970), which we will use extensively. The parameter a_i plays no other role in the analysis.

[11]The quantities of c_i and z_i implied by any feasible t_i and s_i are nonnegative and finite under Assumptions 7 and 8(*ii*) below. See the proof of Theorem 6.

The quantity of capital in region i depends upon the net return there. We define the net return to capital in region i as the following mapping from $X_i \times [0, \bar{s}]$ to the *extended* real numbers

$$NR_i(t_i, s_i) \equiv \begin{cases} (1 - t_i) f_i'(s_i) & \text{if} \quad s_i \in (0, \bar{s}] \\ \lim_{\xi \downarrow 0} \dfrac{(1 - t_i) f_i(\xi)}{\xi} & \text{if} \quad s_i = 0. \end{cases}$$

The limit in the definition of net return is the right derivative of $(1 - t_i) f_i(s_i)$ at zero capital. This limit exists (although it may be infinite) since f_i is concave on $[0, \bar{s}]$ (Rockafellar 1970, Theorem 23.1). Thus, NR_i is well defined. We also know $NR_i(t_i, \cdot)$ is continuous on $[0, \bar{s}]$ for any $t_i \in [0, 1]$ because of the differential continuity properties of concave functions (Rockafellar 1970, Theorem 24.1).[12]

Given any $t \in \mathcal{X}$, an *equilibrium allocation of capital* is any $(s_1, \ldots, s_N)$ such that $s_i \geq 0$ for all i,

$$s_i > 0 \Rightarrow NR_i(t_i, s_i) \geq NR_j(t_j, s_j), \quad \forall i, \ \forall, \ j,$$

and furthermore, if $t \neq (1, \ldots, 1)$ then $\sum_{i=1}^N s_i = \bar{s}$, while if $t = (1, \ldots, 1)$ then $\sum_{i=1}^N s_i = \underline{s}$, where $0 \leq \underline{s} \leq \bar{s}$.

Finally, let $S : \mathcal{X} \to \Re^N$, $S(t) = (S_1(t), \ldots, S_N(t))$, be a function that maps each $t \in \mathcal{X}$ to an equilibrium allocation of capital at t. We define the payoff function for player i to be $u_i(t) \equiv F_i[t_i, S_i(t)]$. Any game $G = (X_i, u_i)_{i=1}^N$ thus defined is a game of fiscal competition for mobile capital.

As constructed above, $G = (X_i, u_i)_{i=1}^N$ is only well defined if the function S exists. Also, for given technology and preferences, there is a different mobile capital game with different payoffs for every S. We are therefore interested in the existence and uniqueness of S.

One case in which there could not be a unique equilibrium allocation of capital is $t = (1, \ldots, 1)$. The net return to capital in every region is zero regardless of the amount of capital there, so every vector $(s_1, \ldots, s_N)$ with $s_i \geq 0$ for all i and $\sum_{i=1}^N s_i = \underline{s}$ is an equilibrium.

The equilibrium allocation of capital is unique in all other cases.

THEOREM 5: *Given the model above and Assumption 7, the function S exists and is unique and continuous on* $\mathcal{X} \setminus \{(1, \ldots, 1)\}$.

We now state and discuss two additional assumptions.

ASSUMPTION 8 (better-reply security): *Assume for all i,*

i. On $(0, \bar{s}]$, $f_i' + s_i f_i'' > 0$.

[12]In particular, if $\lim_{s_i \downarrow 0} f_i'(s_i) = +\infty$, then (a) for $t_i \in [0, 1)$ we have $\lim_{s_i \downarrow 0} NR_i(t_i, s_i) = +\infty = NR_i(t_i, 0)$, and (b) for $t_i = 1$, $\lim_{s_i \downarrow 0} NR_i(1, s_i) = 0 = NR_i(1, 0)$. The interpretation of this case is that the net return in region i with no capital and $t_i < 1$ must exceed the net return in any region with capital, but if instead $t_i = 1$ then the net return is zero.

ii. $\lim_{s_i \downarrow 0} s_i f_i'(s_i) = 0$.

iii. *U_i is continuous on the nonnegative orthant and C^2 with strictly positive first derivatives on the positive orthant.*

Assumption 8(i) requires m_i, the gross payments to capital, to be nondecreasing in the amount of capital. Assumption 8(ii) requires m_i to be continuous at zero.

Together, Assumptions 7 and 8 imply that the game of fiscal competition for mobile capital has all the properties of shared resource games except quasiconcavity. More precisely, Assumptions 1–4 hold, so the game is better-reply secure. The unique degenerate joint strategy is $(1, \ldots, 1)$: all of the sharing rules are discontinuous at this point, and so are all of the payoff functions.[13] As discussed in Section 2.3, this payoff discontinuity is not trivial. It renders useless the standard existence results and cannot be plausibly eliminated by "trimming" the strategy spaces.

ASSUMPTION 9 (quasiconcavity): *Assume for all i,*

i. *U_i is quasiconcave.*

ii. *On* $(0, \bar{s}]$, $s_i f_i' f_i''' - f_i''(f_i' + 2s_i f_i'') \geq 0$.

iii. *On* $(0, \bar{s}]$, $f_i''' \geq 0$, *and* $2(f_i'')^2 - f_i' f_i''' \geq 0$.

iv. $-\infty \leq \lim_{s_i \downarrow 0} f_i''(s_i) < 0$.

Assumptions 9(i) and (ii), together with Assumption 8(i), ensure that $F_i(t_i, s_i)$ is quasiconcave on $[0, 1] \times [0, \bar{s}]$. The remaining parts ensure that either $S_i(\cdot, t_{-i})$ is concave on all of $[0, 1]$, or there is a tax rate $0 < t_i^* < 1$ such that $S_i(\cdot, t_{-i})$ is concave on $[0, t_i^*]$ and equal to zero on $[t_i^*, 1]$. If the former holds, then the payoff function $u_i(\cdot, t_{-i})$ is quasiconcave on $[0, 1]$ by Theorem 1. If the latter holds, then Theorem 1 implies u_i is quasiconcave up to t_i^*. This is good enough, because $S_i(\cdot, t_{-i}) = 0$ on $[t_i^*, 1]$ implies $c_i = z_i = 0$, so $u_i(\cdot, t_{-i}) = U_i(0, 0)$ on $[t_i^*, 1]$. That is to say, the payoff function is flat on $[t_i^*, 1]$. The flat extension of a quasiconcave function is quasiconcave, so again $u_i(\cdot, t_{-i})$ is quasiconcave on $[0, 1]$.

THEOREM 6: *Under Assumptions 7–9, the game of fiscal competition for mobile capital has a pure strategy Nash equilibrium.*

Theorem 6 holds for any finite number of regions, permits general preferences over private and public goods, and does not assume that production technologies have a particular functional form or are identical in all regions. It also does not require $\lim_{\xi \downarrow 0} f_i'(\xi) = +\infty$ for all i (the Inada

[13]The discontinuity is in one's own strategy if $S_i(1, \ldots, 1) < \bar{s}$, otherwise it is the other players' strategies.

condition at zero), which is a condition that many interesting technologies fail.

We can now compare our model and results to those in Bucovetsky (1991) and Laussel–Le Breton (1998). In contrast to those models, we use ad valorem taxes instead of unit taxes. We also assume that the aggregate amount of mobile capital is constant at all tax rates in all regions (although we could vary this at the degenerate joint strategy). The latter assumption is extremely useful for analyzing capital market equilibrium. It is not, however, easily combined with the assumption of unit taxes. Large enough unit tax rates imply a zero equilibrium net return to capital. Even larger rates have undefined effects. One could avoid these problems by requiring the unit tax rate in each region to be less than $f_i'(\bar{s})$. This, however, is quite arbitrary.

Bucovetsky and Laussel–Le Breton handle this problem by assuming that once the equilibrium net return to capital is zero, the capital stock shrinks just enough to keep the net return nonnegative. This is an intuitive solution. The complexities this introduces into analyzing capital market equilibrium, however, make it difficult to establish that payoffs are quasi-concave in each player's own tax rate. Their results are consequently less general.[14]

While we do not use any specific functional forms, we do impose a large number of assumptions on the technologies. It is therefore essential to provide examples that meet these restrictions.

THEOREM 7: *The following production functions satisfy all of the conditions on production functions in Assumptions 7–9:*

i. $f_i(s_i) = (\phi_i - \frac{\beta_i s_i}{2}) s_i$ *provided* $\phi_i > 0$ *and* $0 < \beta_i \leq \frac{\phi_i}{3\bar{s}}$.

ii. $f_i(s_i) = \phi_i \ln(1 + \beta_i s_i)$ *provided* $\phi_i > 0$ *and* $\beta_i > 0$.

iii. $f_i(s_i) = \phi_i[1 - \exp(-\beta_i s_i)]$ *provided* $\phi_i > 0$ *and* $0 < \beta_i < \frac{1}{\bar{s}}$.

Thus, any game of fiscal competition for mobile capital in which all f_i have one or more of these forms and U_i satisfies Assumptions 8(*iii*) and 9(*i*) has a pure strategy Nash equilibrium.

One important production function that does *not* satisfy our assumptions is the Cobb–Douglas, $f_i(s_i) = \phi_i s_i^{\beta_i}$. Specifically, this technology fails

[14]Other differences are as follows. The theorems in Bucovetsky and Laussel–Le Breton hold for just two regions. We permit any number of regions. Bucovetsky assumes that both technologies are quadratic and can differ in just a scale factor. Laussel–Le Breton assume general technologies (they state their result for identical technologies but this is not critical), as do we. Bucovetsky permits general preferences over private and public goods, as do we, while Laussel–Le Breton assume that the private and public goods are perfect substitutes. Bucovetsky closes his model by assuming that the owners of fixed factors also own an equal share of the total capital stock. In contrast, Laussel–Le Breton and we assume that the owners of capital are essentially absentee. We can take Bucovetsky's approach, however, if we are willing to assume the Inada condition at zero. This is the subject of ongoing work.

Assumption 9(iii), since $2(f_i'')^2 - f_i' f_i''' = \beta_i - 1 < 0$. If we assume that all of the production functions have this form and $\beta_i = \beta$ for all i, however, we can solve for S_i explicitly. We can then show that $u_i(\cdot, t_{-i})$ is quasiconcave directly.

THEOREM 8: *The production function $f_i(s_i) = \phi_i s_i^{\beta}$ with $\phi_i > 0$ and $0 < \beta < 1$ satisfies all parts of Assumptions 7–9 except Assumption 9(iii). Nevertheless, if* all *production functions take this form and U_i satisfies Assumptions 8(iii) and 9(i), then the game of fiscal competition for mobile capital has a pure strategy Nash equilibrium.*

Theorems 6–8 together suggest that the Cobb–Douglas production function is somewhat misleading for the *theoretical* analysis of existence in models of fiscal competition. While it is convenient for computational purposes since all regions with this technology must have capital in equilibrium, it presents significant, but apparently idiosyncratic, problems in regards to showing quasiconcavity. Determining precisely how idiosyncratic these problems are would require further research.

6. Conclusion

An analysis of equilibrium existence should shed some light on the basic structure of the games to which it applies. Our analysis of games of fiscal competition suggests that they fall into two broad groups. In one group are those where the owners of mobile factors derive no benefit from the goods provided by local government. The canonical game of fiscal competition for mobile capital belongs in this group. For this reason, lower taxes always attract capital to a region and higher taxes always drive it away. We present general conditions on utility and production functions that ensure that a pure strategy Nash equilibrium exists in this game. There is just a single discontinuity point in that game, but it is sufficient to render useless the standard existence theorems. It is also fundamental, in the sense that it emerges from the basic incentives in the game, and simple modifications of the game are inadequate.

The other group of fiscal competition games are those in which the owners of mobile factors do benefit from publicly provided goods. Games in which the mobile factor is labor or the publicly provided good is infrastructure belong in this group. In some of these games, lower taxes may attract or repel the mobile factor depending on the initial level of taxes. The set of discontinuity points in those games is likely to be much larger, as shown in the extended example in Rothstein (2005).

Both groups of fiscal competition games may be modeled as games in which payoffs have two components. One component captures the direct effect of one's strategy on utility, holding fixed the amount of shared resource one obtains. The other captures the indirect effect, which operates through the amount of shared resource one obtains and through this depends on the strategies of the other players. We call games in which payoffs satisfy these conditions, certain additional ones at discontinuity points, and quasiconcavity, shared resource games. We derive a pure strategy Nash equilibrium existence theorem for these games. This result requires the strongest result in the recent work by Reny (1999). Reny's result, however, requires the analysis of the closure of a high-dimensional object. Our general theorem does not. We also give an independent analysis of quasiconcavity in these games.

Finally, it has recently been argued that the mobile capital game studied here and in Laussel–Le Breton is too simple (Bayindir-Upmann and Ziad 2005). The basic objection is that the owners of fixed factors should also own the mobile capital, so the model is fully closed (as in Bucovetsky 1991). Closing the model in this way defines a game in the second group. As capital owners the players prefer zero taxes, since the productivity of capital is unaffected by public consumption goods. As consumers, however, they desire positive taxes, since they derive utility from public goods. Preliminary work shows that our general theorem and analysis of quasiconcavity is directly applicable to this model as well, provided we are willing to assume the Inada condition at zero. This is the subject of ongoing work.

Appendix

Proof of Theorem 1: Define the following notation for convex combinations of any points x and x'.

$$x\lambda x' \equiv \lambda x + (1-\lambda)x'$$

Fix $x_{-i} \in \mathcal{X}_{-i}$, x_i, and x_i' in X_i, and $\lambda \in (0, 1)$. Then,

$$\begin{aligned} F_i[x_i\lambda x_i', S_i(x_i\lambda x_i', x_{-i})] &\geq F_i[x_i\lambda x_i', S_i(x_i, x_{-i})\lambda S_i(x_i', x_{-i})] \\ &\geq \min\{F_i[x_i, S_i(x_i, x_{-i})], F_i[x_i', S_i(x_i', x_{-i})]\}. \end{aligned}$$

The first inequality uses both the fact that $S_i(\cdot, x_{-i})$ is concave and F_i is nondecreasing in s_i. The second uses the quasiconcavity of F_i. ■

Proof of Theorem 2: Fix $x_{-i} \in \mathcal{X}_{-i}$, x_i and x_i' in $\mathrm{ri}(X_i)$, and $\lambda \in (0, 1)$. Suppose $S_i(\cdot, x_{-i})$ is continuous. Assumption 6(ii) applies. Define $\hat{s}_i = T[S_i(x_i, x_{-i})]$, and $\hat{s}_i' = T[S_i(x_i'), x_{-i})]$. We now have (see above for the notation)

$$\begin{aligned} u_i(x_i\lambda x_i', x_{-i}) &= F_i[x_i\lambda x_i', S_i(x_i\lambda x_i', x_{-i})] \\ &= F_i\big[x_i\lambda x_i', T^{-1}\{T[S_i(x_i\lambda x_i', x_{-i})]\}\big] \\ &\geq F_i\big[x_i\lambda x_i', T^{-1}\{T[S_i(x_i, x_{-i})]\lambda T[S_i(x_i', x_{-i})]\}\big] \\ &= F_i\big[x_i\lambda x_i', T^{-1}(\hat{s}_i\lambda\hat{s}_i')\big]. \end{aligned}$$

The inequality uses the fact that by assumption $T[S_i(\cdot, x_{-i})]$ is concave, T^{-1} must be increasing, and F_i is nondecreasing in s_i. It now follows from Assumption 6(*ii*)(*b*) that $u_i(\cdot, x_{-i})$ is quasiconcave on $\mathrm{ri}(X_i)$. It is also continuous, since F_i and $S_i(\cdot, x_{-i})$ are continuous. We can therefore extend quasiconcavity to the boundary and conclude that $u_i(\cdot, x_{-i})$ is quasiconcave.

Now suppose $S_i(\cdot, x_{-i})$ is not continuous. Assumption 6(*iii*) applies, so $S_i(\cdot, x_{-i})$ is concave. F_i is quasiconcave on $X_i \times [0, \bar{s}]$ by both Assumption 6(*i*) and continuity. It is nondecreasing in s_i by Assumption 4(*ii*). $u_i(\cdot, x_{-i})$ is therefore quasiconcave by Theorem 1.

Finally, any discontinuity points of $S_i(\cdot, x_{-i})$ must lie on the relative boundary of X_i by Rockafellar (1970), Theorem 10.1. ■

Proof of Theorem 3: To see that the game is compact, note that X_i has the required properties by Assumption 1, and u_i is bounded since F_i is continuous and real valued with compact domain. The game is quasiconcave by Assumptions 1 and 5. Thus, the only issue is better-reply security.

We need two preliminary results. The first identifies certain payoffs that an agent can achieve by unilaterally deviating from degenerate joint strategies. ■

LEMMA 1: *Fix any agent i and any* $(\bar{x}_i, \bar{x}_{-i}) \in \Delta$. *Suppose there exists some* $x_i \in X_i$, s_i *satisfying* $\tilde{s}_i \geq s_i \geq 0$, *and* $\alpha \in \Re$ *such that* $F_i(x_i, s_i) > \alpha$. *Then there exists* $\bar{x}_i' \in X_i$ *such that* $(\bar{x}_i', \bar{x}_{-i}) \in \mathcal{X} \backslash \mathcal{D}_i$, *and* $u_i(\bar{x}_i', \bar{x}_{-i}) > \alpha$.

Proof of Lemma 1: F_i is continuous in its first argument, so there exists a neighborhood of x_i, say $\mathcal{N}(x_i)$, such that

$$F_i(x_i', s_i) > \alpha, \quad \forall x_i' \in \mathcal{N}(x_i).$$

If $(x_i, \bar{x}_{-i}) \in \mathcal{X} \backslash \mathcal{D}_i$, then define $\bar{x}_i' = x_i$. If $(x_i, \bar{x}_{-i}) \in \mathcal{D}_i$, then $\mathcal{N}(x_i)$ must contain a point $\bar{x}_i'$ such that $(\bar{x}_i', \bar{x}_{-i}) \in \mathcal{X} \backslash \mathcal{D}_i$ by Assumption 2(*iii*). In both cases, we have $F_i(\bar{x}_i', s_i) > \alpha$, and $(\bar{x}_i', \bar{x}_{-i}) \in \mathcal{X} \backslash \mathcal{D}_i$. Therefore $S_i(\bar{x}_i', \bar{x}_{-i}) \geq \tilde{s}_i$ by Assumption 2(*iv*). The premise $\tilde{s}_i \geq s_i$ then gives $S_i(\bar{x}_i', \bar{x}_{-i}) \geq s_i$. F_i is nondecreasing in shared resource so $u_i(\bar{x}_i', \bar{x}_{-i}) = F_i[\bar{x}_i', S_i(\bar{x}_i', \bar{x}_{-i})] \geq F_i(\bar{x}_i', s_i) > \alpha$. ■

Our next result is obvious, but we include it because the idea of securing a payoff is probably unfamiliar. We show that if a player *achieves* more than a given payoff at a continuity point then she can also *secure* more than this payoff at that point. Formally,

LEMMA 2: *Suppose* $u_i(x) > \alpha$ *at some* $x \in \mathcal{X} \backslash \mathcal{D}_i$, *and* $\alpha \in \Re$. *Then i can secure a payoff at x that is strictly above* α.

Proof of Lemma 2: u_i is continuous at x since S_i is continuous at x. Define $\epsilon \equiv \frac{u_i(x)-\alpha}{2} > 0$. Then there is a neighborhood of x_{-i}, say $\mathcal{N}(x_{-i})$, such that $x'_{-i} \in \mathcal{N}(x_{-i})$ implies $u_i(x) - u_i(x_i, x'_{-i}) < \epsilon$. Rearranging gives

$$u_i(x_i, x'_{-i}) > \frac{u_i(x)+\alpha}{2}, \quad \forall x'_{-i} \in \mathcal{N}(x_{-i}).$$

Player i therefore secures $\frac{u_i(x)+\alpha}{2}$ at x. This is strictly above α since $u_i(x) > \alpha$. ■

LEMMA 3: *The shared resource game is better-reply secure.*

Proof of Lemma 3: Fix (x^*, u^*) in the closure of the graph of the vector payoff function. By definition there is a sequence lying entirely in the graph of the vector payoff function converging to this point. Denote this sequence as

$$\left\{\left(x_1^j, \ldots, x_N^j, u_1(x^j), \ldots, u_N(x^j)\right)\right\}_{j=1}^{\infty}.$$

Each component converges to its respective component in (x^*, u^*), so

$$u_i^* = \lim_{j\to\infty} u_i(x^j), \quad \forall i.$$

Since $\mathcal{X}$ is compact we know $x^* \in \mathcal{X}$ and $u_i(x^*)$ is well defined for all i. Possible discontinuities in the payoff functions make it possible that $\lim_{j\to\infty} u_i(x^j) \neq u_i(x^*)$.

Define $\xi_i(s_i)$ to be any optimal choice of x_i given a fixed quantity of shared resource s_i. Formally,

$$\xi_i(s_i) \in \arg\max_{x_i \in X_i} F_i(x_i, s_i). \tag{A1}$$

The set is nonempty given any s_i, since F_i is continuous, and X_i is compact.

Suppose first that for all agents, $u_i(x^*) = u_i^*$. By assumption x^* is not an equilibrium point, so some agent k has a strategy $\bar{x}_k$ such that $u_k(\bar{x}_k, x^*_{-k}) > u_k(x^*)$. Therefore $u_k(\bar{x}_k, x^*_{-k}) > u_k^*$. If $(\bar{x}_k, x^*_{-k}) \in \mathcal{X}\backslash\mathcal{D}_k$, then k can secure a payoff at x^* that is strictly above u_k^* by Lemma 2. If $(\bar{x}_k, x^*_{-k}) \in \mathcal{D}_k$, then note that $F_k[\bar{x}_k, S_k(\bar{x}_k, x^*_{-k})] = u_k(\bar{x}_k, x^*_{-k}) > u_k^*$, and $S_k(\bar{x}_k, x^*_{-k}) \leq \tilde{s}_k$ by Assumption 2(*iv*), so Lemma 1 applies. This gives $\bar{x}'_k$ such that $(\bar{x}'_k, x^*_{-k}) \in \mathcal{X}\backslash\mathcal{D}_k$, and $u_k(\bar{x}'_k, x^*_{-k}) > u_k^*$. Again, k can secure a payoff at x^* that is strictly above u_k^* by Lemma 2.

Now suppose instead that for some agent i, $u_i(x^*) \neq u_i^*$. The payoff function for this agent is discontinuous at this point, so S_i must be discontinuous at this point.[15] It follows that $x^* \in \mathcal{D}_i \subset \Delta$. Now, suppose we can show that for some (possibly different) agent k,

[15]The converse is not true—u_i may be continuous where S_i is discontinuous—and this fact rules out certain approaches to the proof.

$$F_k[\xi_k(\bar{s}/N), \bar{s}/N] \geq u_k^*. \tag{A2}$$

That is to say, player k's limit payoff is less than or equal to the maximum utility she could achieve if she were assigned an equal share of the shared resource. We have $F_k[\xi_k(\tilde{s}_k), \tilde{s}_k] > F_k[\xi_k(\bar{s}/N), \bar{s}/N]$ from Assumption $4(iii)$, so $F_k[\xi_k(\tilde{s}_k), \tilde{s}_k] > u_k^*$. Since $x^* \in \Delta$, we have $S_k(x^*) \leq \tilde{s}_k$ by Assumption $2(iv)$.[16] We can therefore use Lemma 1. As above, this gives $\bar{x}_k'$ such that $(\bar{x}_k', x_{-k}^*) \in \mathcal{X}_k \backslash \mathcal{D}_k$, and $u_k(\bar{x}_k', x_{-k}^*) > u_k^*$. Player k can then secure a payoff at x^* that is strictly above u_k^* by Lemma 2. This would complete the proof.

So, suppose (A2) fails to hold for all agents:

$$u_i^* > F_i[\xi_i(\bar{s}/N), \bar{s}/N], \quad \forall i. \tag{A3}$$

It follows from convergence that $u_i(x^j)$ can be made arbitrarily close to u_i^* for all j sufficiently large. Thus, for each individual i, there is an integer, say j_i, such that

$$u_i(x^j) > F_i[\xi_i(\bar{s}/N), \bar{s}/N], \quad \forall i, \forall j \geq j_i. \tag{A4}$$

This says that at any term high enough in the sequence, the payoff for i must strictly exceed the best she could do if she were guaranteed an equal share of the resource. We now show that in order to realize this payoff, she must in fact have strictly more than an equal share of the resource:

$$S_i(x^j) > \bar{s}/N, \quad \forall i, \forall j \geq j_i. \tag{A5}$$

If (5) does not hold, then for some term $j \geq j_i$ we have $S_i(x^j) \leq \bar{s}/N$. Then $F_i[\xi_i(\bar{s}/N), \bar{s}/N] \geq F_i(x_i^j, \bar{s}/N) \geq F_i[x_i^j, S_i(x^j)] = u_i(x^j)$. This would contradict (A4). Therefore (A5) holds.

Finally, define $\bar{j} = \max\{j_1, \ldots j_N\}$. It follows from (5) that the inequality $S_i(x^{\bar{j}}) > \bar{s}/N$ holds for all i. Therefore $\sum_{i=1}^N S_i(x^{\bar{j}}) > N(\bar{s}/N) = \bar{s}$. This violates both aggregate resource constraints, yet one of them must hold at $x^{\bar{j}}$. The contradiction means (A3) does not hold and so (A2) holds for some agent. ■

Proof of Theorem 4: Let x^* be an equilibrium joint strategy. Suppose by way of contradiction that $x^* \in \Delta$. Some individual i must have $S_i(x^*) \leq \underline{s}/N$, otherwise $\sum_{i=1}^N S_i(x^*) > N\underline{s}/N = \underline{s}$. This would contradict Assumption $2(ii)$. Therefore, for some i,

$$\begin{aligned} F_i[\xi_i(\tilde{s}_i), \tilde{s}_i)] &> F_i[\xi_i(\bar{s}/N), \bar{s}/N] \geq F_i\big(x_i^*, \bar{s}/N\big) \\ &\geq F_i\big(x_i^*, \underline{s}/N\big) \geq F_i\big[x_i^*, S_i(x^*)\big] = u_i(x^*). \end{aligned}$$

[16]This is where we use the fact Assumption $2(iv)$ holds for each player on all of Δ. We need this because, while we know $x^* \in \Delta$, we do not know whether $x^* \in \mathcal{D}_k$, or even if this set is nonempty.

The first inequality uses Assumption 4(iii). We can now apply Lemma 1 of Theorem 3 with the assignments $x_i = \xi_i(\tilde{s}_i)$, $s_i = \tilde{s}_i$, and $\alpha = u_i(x^*)$. It follows that there exists $\bar{x}_i' \in X_i$ such that $u_i(\bar{x}_i', x_{-i}^*) > u_i(x^*)$. This means player i has a better reply, a contradiction. ■

Supplementary example. Suppose $N = 3$, $X_i = [0, 1]$, $\bar{s} = 1$, and

$$S_1(x_1, x_2, x_3) = \frac{4}{10},$$

$$S_2(x_1, x_2, x_3) = \begin{cases} \left[\dfrac{x_2(1 - x_3)}{x_2(1 - x_3) + x_3}\right]\dfrac{6}{10}, & \text{otherwise} \\ \dfrac{3}{10}, & x_1 \in [0, 1],\ x_2 = 0, \text{and } x_3 = 0, \end{cases}$$

$$S_3(x_1, x_2, x_3) = \begin{cases} \left[\dfrac{x_3}{x_2(1 - x_3) + x_3}\right]\dfrac{6}{10}, & \text{otherwise} \\ \dfrac{3}{10}, & x_1 \in [0, 1],\ x_2 = 0, \text{and } x_3 = 0. \end{cases}$$

Define $F_i(x_i, s_i) = x_i s_i$, and $u_i(x) = F_i[x_i, S_i(x_i, x_{-i})]$.

$G = (X_i, u_i)_{i=1}^N$ is a shared resource game. We have $\mathcal{D}_1 = \emptyset$, $\mathcal{D}_2 = \mathcal{D}_3 = \Delta = \{(x_1, x_2, x_3) \mid x_1 \in X_1,\ x_2 = 0,\ x_3 = 0\}$. For Assumption 2, define $\underline{s} = 1$, and $\tilde{s}_i = 4/10 > 1/3 = \bar{s}/N$ for all three players. The only issue is Assumption 2(iv). The inequalities hold trivially for Player 1. For Player 2 and any $(x_1, x_2, x_3) \in \Delta$, we have $S_2(x_1, x_2, x_3) = 3/10 < \tilde{s}_2$. A deviation to any $x_2' \in (0, 1]$ gives $S_2(x_1, x_2', x_3) = 6/10 > \tilde{s}_2$. The same analysis holds for Player 3. F_i is clearly quasiconcave and each $S_i(\cdot, x_{-i})$ is concave, so Theorem 1 gives $u_i(\cdot, x_{-i})$ concave.

This example establishes the following. First, a sharing rule can have a lower value at a continuity point than at a discontinuity point. For example, $S_2(1, .5, 1) = 0 < 3/10 = S_2(1, 0, 0)$, yet $(1, .5, 1) \in \mathcal{X} \backslash \mathcal{D}_2$, and $(1, 0, 0) \in \mathcal{D}_2$. This does not violate Assumption 2(iv) because no point of the form $(1, \cdot, 1)$ belongs to Δ, so $(1, .5, 1)$ cannot result from a unilateral deviation by Player 2 from a point in Δ. Second, while a degenerate joint strategy cannot be an equilibrium, an equilibrium can contain a strategy that is part of some degenerate joint strategy. A joint strategy in which x_1 takes any value, $x_2 = 0$, and $x_3 = 1$ is a Nash equilibrium. If instead we had $x_3 = 0$ then this would be a degenerate joint strategy.

Proof of Theorem 5: We first establish a lemma that we will use here and in the proof of Theorem 6. ■

LEMMA 4: *Suppose Assumption 7 holds,* $t \in \mathcal{X}\backslash\{(1, \ldots, 1)\}$, *and* $(s_1, \ldots, s_N)$ *is an equilibrium allocation of capital. Then*

$$s_i > 0 \Longrightarrow t_i < 1.$$

Proof of Lemma 4: Suppose instead $t_i = 1$. The definitions of equilibrium and net return, respectively, imply $NR_i(t_i, s_i) \geq NR_j(t_j, s_j)$ for all j and $NR_i(t_i, s_i) = 0$, so $NR_j(t_j, s_j) = 0$ for all j. Since $t \in \mathcal{X}\backslash\{(1, \ldots, 1)\}$, there exists some region $k \neq i$ such that $t_k < 1$. We have $NR_k(t_k, s_k) > 0$ for any $s_k \in (0, \bar{s}]$, it is decreasing in S_k, and it is continuous at zero, so $NR_k(t_k, 0) > 0$. This is a contradiction.

We now prove Theorem 5. We first show that if an equilibrium allocation of capital exists at $t \in \mathcal{X}\backslash\{(1, \ldots, 1)\}$ then it must be unique. This relies on the strict concavity of each f_i. We then show that a solution to a particular optimum problem exists and defines an upper semicontinuous correspondence mapping tax vectors to equilibria. It follows from the uniqueness of equilibrium that this correspondence is a continuous function.

Let $\tilde{S} : \mathcal{X}\backslash\{(1, \ldots, 1)\} \to \tilde{\Xi} \subset \Re^N$ denote the mapping from each tax vector to the set of equilibrium allocations of capital. Suppose that at some t there are distinct vectors $(s_1, \ldots, s_N)$ and $(s'_1, \ldots, s'_N)$ in $\tilde{S}(t)$; in particular, $s_j > s'_j \geq 0$ for some j. The total amount of capital in both equilibria is $\bar{s}$. Thus, there must be a region $k \neq j$ such that $s'_k > s_k \geq 0$. We have from the definition of equilibrium that $NR_j(t_j, s_j) \geq NR_k(t_k, s_k)$ and $NR_k(t_k, s'_k) \geq NR_j(t_j, s'_j)$. From Lemma 4, we have $t_j < 1$ and $t_k < 1$. Under Assumption 7, f'_j and f'_k are always decreasing. It follows that $NR_j(t_j, s'_j) > NR_j(t_j, s_j)$ and $NR_k(t_k, s_k) > NR_k(t_k, s'_k)$. Combining the inequalities gives

$$NR_j(t_j, s'_j) > NR_j(t_j, s_j) \geq NR_k(t_k, s_k) > NR_k(t_k, s'_k) \geq NR_j(t_j, s'_j),$$

a contradiction. It follows that each set $\tilde{S}(t)$ either is empty or has just one element.

Now consider the problem of maximizing total net output. Define $\Sigma = \times_{i=1}^{N}[0, \bar{s}]$, the function $f : \mathcal{X}\backslash\{(1, \ldots, 1)\} \times \Sigma \to \Re$, where $f(t, \hat{s}) = \sum_{i=1}^{N}(1 - t_i) f_i(\hat{s}_i)$, the set $\Psi = \{\hat{s} \in \Re^N \mid \hat{s}_i \geq 0,\ i = 1, \ldots, N,\ \sum_{i=1}^{N} \hat{s}_i = \bar{s}\}$, and the mapping $\hat{S} : \mathcal{X}\backslash\{(1, \ldots, 1)\} \to \hat{\Xi} \subset \Re^N$, where $\hat{S}(t) = \arg\max\{f(t, \hat{s}) \mid \hat{s} \in \Psi\}$. The set Ψ is compact and f is continuous by Assumption 7. It follows that $\hat{S}(t) \neq \emptyset$ and $\hat{S}$ is an upper semicontinuous correspondence, by the theorem of the maximum (Sundaram 1996, Theorem 9.14).

Fix any $t \in \mathcal{X}\backslash\{(1, \ldots, 1)\}$ and any $\hat{s} = (\hat{s}_1, \ldots, \hat{s}_N) \in \hat{S}(t)$. We now characterize $\hat{s}$ in terms of the net return to capital. This will allow us to show that it is an equilibrium, so $\hat{S}(t) \subset \tilde{S}(t)$.

Renumber the regions so that the first M regions are those with tax rates less than 1, so $i \leq M \Longleftrightarrow t_i < 1$. At least one such region exists. Regions numbered above M have tax rates equal to 1, so $M < i \leq N \Longleftrightarrow t_i = 1$ (if no such regions exist then $M = N$).

As a preliminary result, we show that no region numbered above M has any capital:[17]

$$M < i \leq N \Longrightarrow \hat{s}_i = 0. \tag{A6}$$

Suppose instead $\hat{s}_i > 0$. We can remove the $\hat{s}_i$ units of capital from region i with no reduction in f (since $t_i = 1$) and add them to the capital in any region $j \leq M$ and increase f (since $f'_j > 0$ by Assumption 7(*ii*)). Therefore the new allocation of capital, $\hat{s}'$, satisfies both $f(t, \hat{s}') > f(t, \hat{s})$ and $\hat{s}' \in \Psi$. This contradicts $\hat{s} \in \hat{S}(t)$.

We now use a version of the Kuhn–Tucker theorem that does not require differentiability at the origin for each f_i (Rockafellar 1970, Theorem 28.3 and Corollary 28.3.1). The details are available on request, but the final result is completely straightforward. All of the regions with capital have the same net return, and this is greater than or equal to the net return in all of the regions without capital. In notation, let λ denote the multiplier on the aggregate resource constraint $\sum_{i=1}^{M} \hat{s}_i = \bar{s}$. The conditions are, if $\hat{s}_i > 0$ then $NR_i(t_i, \hat{s}_i) = (1 - t_i) f'(\hat{s}_i) = \lambda$, and if $\hat{s}_i = 0$ then $NR_i(t_i, \hat{s}_i) = \lim_{\xi \downarrow 0} \frac{(1 - t_i) f_i(\xi)}{\xi} \leq \lambda$.

To see that $\hat{s}$ is an equilibrium allocation of capital, note first that it satisfies nonnegativity and the adding-up condition by construction. Now fix any region i with $\hat{s}_i > 0$ and any other region j. By the previous result, $NR_i(t_i, \hat{s}_i) = \lambda$. We also know that either $\hat{s}_j = 0$ and $NR_j(t_j, \hat{s}_j) \leq \lambda$, or $\hat{s}_j > 0$ and $NR_j(t_j, \hat{s}_j) = \lambda$. In either case, $NR_j(t_j, \hat{s}_j) \leq \lambda$. Therefore $NR_i(t_i, \hat{s}_i) = \lambda \geq NR_j(t_j, \hat{s}_j)$.

We conclude that $\emptyset \neq \hat{S}(t) \subset \tilde{S}(t)$ at any $t \in \mathcal{X} \setminus \{(1, \ldots, 1)\}$, so $\tilde{S}(t) \neq \emptyset$. It follows from our first result that $\tilde{S}$ is single valued. This with $\hat{S}(t) \subset \tilde{S}(t)$ gives $\hat{S} = \tilde{S}$. $\hat{S}$ is therefore single valued, so it is a continuous function. Finally, $S = \tilde{S}$ from the definitions and single valuedness of $\tilde{S}$. Thus, S exists and is unique and continuous. ■

Proof of Theorem 6: Assumption 1 is immediate.

We now show that the equilibrium allocation of capital functions, $S_1(t), \ldots, S_N(t)$, satisfy Assumption 2. It is straightforward to show that each S_i is discontinuous at $(1, \ldots, 1)$ (although not necessarily in i's own strategy). Theorem 5 establishes continuity at all other tax vectors, so $\{(1, \ldots, 1)\} = \mathcal{D}_i = \Delta$. Assumptions 2(*i*)–2(*ii*) hold by construction. For Assumption 2(*iii*), simply note that any unilateral deviation from $(1, \ldots, 1)$ by player i gives a continuity point of S_i. Now fix $\tilde{s}_i = \bar{s}$

[17]Note well that the converse of (A6) is not true.

for all i . Immediately $\tilde{s}_i \geq S_i(1, \ldots, 1)$. Fix any vector $(1, \ldots, t_i, \ldots, 1)$ with $t_i < 1$. It follows from Lemma 4 that $S_i(1, \ldots, t_i, \ldots, 1) = \bar{s}$, since $S_j(1, \ldots, t_i, \ldots, 1) = 0$ for any player j different from i. Therefore $S_i(1, \ldots, t_i, \ldots, 1) = \tilde{s}_i \geq S_i(1, \ldots, 1)$. This establishes Assumption 2(iv).

Assumption 3 holds by definition as long as F_i in the text is well defined. The only issue is whether $f_i(s_i) - m_i(s_i)$ is always nonnegative. We have $f_i(0) - m_i(0) = 0$, it is continuous at zero (given Assumption 8(ii)) and increasing at all $s_i > 0$ (since $f_i'' < 0$), so this follows.

Assumption 4(i) holds since $t_i m_i(s_i)$ and $f_i(s_i) - m_i(s_i)$ are both continuous on $X_i \times [0, \bar{s}]$ and U_i is continuous. For Assumption 4(ii), suppose $s_i \in (0, \bar{s}]$ and $t_i \in (0, 1]$. Then $c_i > 0$, $z_i > 0$, and $\frac{\partial F_i}{\partial s_i} = (\frac{\partial U_i}{\partial c_i})(-s_i f_i'') + (\frac{\partial U_i}{\partial z_i})[t_i(f_i' + s_i f_i'')] > 0$ given Assumption 8(i). So $F_i(t_i, \cdot)$ is increasing on $(0, \bar{s}]$ for any $t_i \in (0, 1]$. With continuity it is nondecreasing on $[0, \bar{s}]$ for any $t_i \in [0, 1]$. For Assumption 4(iii), fix any $s_i \in [\bar{s}/N, \bar{s}]$ and $t_i \in (0, 1)$. We have $\frac{\partial F_i}{\partial t_i} = (\frac{\partial U_i}{\partial z_i}) s_i f_i' > 0$. This with continuity gives $\max_{t_i \in X_i} F_i(t_i, s_i) = F_i(1, s_i)$ and $\max_{t_i \in X_i} F_i(t_i, \bar{s}/N) = F_i(1, \bar{s}/N)$. We have $F_i(1, s_i) > F_i(1, \bar{s}/N)$ from the previous inequality.

We now turn to quasiconcavity. The game is well defined, so we have a capital market equilibrium mapping $(S_1(t), \ldots, S_N(t))$. The numbering of regions is arbitrary, so without any loss of generality the entire analysis is in terms of region 1. Fix any tax vector $t_{-1} \in \mathcal{X}_{-1}$.

Suppose first that $t_{-1} = (1, \ldots, 1)$. Note first that for any $t_1 \in [0, 1]$, we have $F_1(t_1, \bar{s}) = U_1[f_1(\bar{s}) - \bar{s} f_1'(\bar{s}),\ t_1 \bar{s} f_1'(\bar{s})]$. This is increasing in t_1. For $t_1 \in [0, 1)$ we have $S_1(t_1, t_{-1}) = \bar{s}$, so $u_1(t_1, t_{-1}) = F_1(t_1, \bar{s})$. Necessarily $S_1(1, t_{-1}) \leq \bar{s}$, so $u_1(1, t_{-1}) = F_1[1, S_1(1, t_{-1})] \leq F_1(1, \bar{s})$. It follows that $u_1(\cdot, t_{-1})$ either increases on all of $[0, 1]$ or jumps down at 1, and in either case it is quasiconcave.

For the remainder of the proof, we suppose that $t_{-1} \neq (1, \ldots, 1)$.

We first show F_1 is quasiconcave. At any point in $(t_1, s_1) \in (0, 1) \times (0, \bar{s})$, a sufficient condition for $2\frac{\partial F_1}{\partial t_1}\frac{\partial F_1}{\partial s_1}\frac{\partial^2 F_1}{\partial t_1 \partial s_1} - (\frac{\partial F_1}{\partial t_1})^2\frac{\partial^2 F_1}{\partial s_1^2} - (\frac{\partial F_1}{\partial s_1})^2\frac{\partial^2 F_1}{\partial t_1^2} \geq 0$ is

$$s_1 \frac{\partial U_1}{\partial c_1}[s_1 f_1' f_1''' - f_1''(f_1' + 2 s_1 f_1'')]$$
$$+ t_1 \frac{\partial U_1}{\partial z_1}\left[2(f_1' + s_1 f_1'')^2 - (s_1 f_1')(2 f_1'' + s_1 f_1''')\right] \geq 0.$$

This holds if the bracketed terms are both nonnegative. The first holds by Assumption 9(ii). The second can be rearranged into $2 f_1'[f_1' + s_1 f_1''] + s_1^2[2(f_1'')^2 - f_1' f_1''']$. This is nonnegative by Assumption 8(i) and Assumption 9(iii). F_1 is then quasiconcave (on $[0, 1] \times [0, \bar{s}]$) by continuity.

We now turn to a detailed analysis of the concavity of S_1. Not assuming the Inada condition at zero accounts for much of the complexity of the analysis. The Inada condition at zero would guarantee that, for all values of $t_1 \in (0, 1)$, region 1 has capital in equilibrium and so does every other region that has a tax rate less than 1. Without the Inada condition, however, a region can switch from having zero capital to having a positive amount of capital as the tax rate in region 1 increases. This means that the equal net return condition does not define a fixed set of equations characterizing the capital market equilibrium. This in turn means we cannot use the implicit function theorem for a global analysis of the concavity of $S_1(\cdot, x_{-1})$. It also means that S_1 need not be globally differentiable, despite the differentiability of the production functions.

The analysis proceeds in four steps as follows:

i. We first show that if t_1 increases, then for all regions $j \geq 2$, S_j cannot decrease. Since the total amount of capital is fixed, S_1 cannot increase.[18]

ii. This allows us to partition $X_1 = [0, 1]$ into numbered convex components on which the number of regions with capital (exclusive of region 1) is constant on each component and increasing in the number of the component.

iii. We then use the implicit function theorem to show that S_1 is concave on each component (up to any point where it equals zero).

iv. The final step is to show that S_1 becomes (nondifferentiably) steeper at points where two components join.

With S_1 nonincreasing and concave on each component and steeper at points where any two components join, it must be concave (again, up to any point where it equals zero).

Step (i). Define the vectors $t = (t_1, t_{-1})$ and $t' = (t'_1, t_{-1})$ with $t'_1 > t_1$, and the associated capital market equilibria $(S_1(t), \ldots, S_N(t))$ and $(S_1(t'), \ldots, S_N(t'))$. Suppose by way of contradiction that there is some region $j \geq 2$ with $0 \leq S_j(t') < S_j(t)$. The total amount of capital in both equilibria is $\bar{s}$, since $t_{-1} \neq (1, \ldots, 1)$. Thus, the capital that leaves region j must move to at least one other region $i \neq j$ (nothing so far rules out $i = 1$). So there exists a region i such that $S_i(t') > S_i(t) \geq 0$. It follows that $S_j(t) > 0$ and $S_i(t') > 0$. These inequalities and the definition of equilibrium give $NR_j[t_j, S_j(t)] \geq NR_i[t_i, S_i(t)]$ and $NR_i[t'_i, S_i(t')] \geq NR_j[t'_j, S_j(t')]$. The inequalities and Lemma 4 give $t_j < 1$ and $t'_i < 1$. By construction, we have $t'_j = t_j$ and $t_i \leq t'_i$, so we also have $t'_j < 1$ and $t_i < 1$. Finally, under Assumption 7, f'_j and f'_i are always decreasing. It now follows that

[18]If we had differentiability, these would be the familiar results $\frac{\partial S_j}{\partial t_1} \geq 0$ and $\frac{\partial S_1}{\partial t_1} \leq 0$. See Wildasin (1988).

$$NR_j[t'_j, S_j(t')] = NR_j[t_j, S_j(t')] > NR_j[t_j, S_j(t)] \geq NR_i[t_i, S_i(t)]$$

and

$$NR_i[t_i, S_i(t)] \geq NR_i[t'_i, S_i(t)] > NR_i[t'_i, S_i(t')] \geq NR_j[t'_j, S_j(t')].$$

These are inconsistent.

From this point on we treat t_{-1} as a constant and suppress all references to it.

Step (ii). We say *region i is active at t_1* if $S_i(t_1) > 0$. The *number of active regions at t_1 exclusive of region 1* is given by the function $\rho : [0, 1] \rightarrow \{0, 1, \ldots, N-1\}$,

$$\rho(t_1) = \#\{j \mid S_j(t_1) > 0, \quad j = 2, \ldots, N\}.$$

Let M denotes the number of distinct values that $\rho(\cdot)$ takes on [0, 1] and the values themselves by $r^1 < \cdots < r^M$. Thus, r^k denote a particular number of active regions. Let I^k be the set of points at which exactly r^k regions are active:

$$I^k = \{t_1 \mid \rho(t_1) = r^k\}, \quad k = 1, \ldots, M.$$

The collection of sets $I^1, \ldots, I^M$ clearly forms a partition of [0, 1]. We need to establish the basic properties of this partition.

First, if a region is active (or inactive) at $t_1 \in I^k$ then it is active (inactive) at *all* points in I^k, in which case we say it is *active (inactive) on* I^k. Suppose not, so some region j is inactive at $t_1 \in I^k$ and active at $t'_1 \in I^k$. By construction, $\rho(t_1) = \rho(t'_1)$, so some other region $j' \geq 2$ must be active at t_1 and inactive at t'_1. One of these regions switches from active to inactive as t_1 increases, a contradiction. Thus r^k actually denotes the number of regions (other than 1) that are active on all of I^k.

Secondly, each I^k is a convex set. We know that $\rho(\cdot)$ is nondecreasing, which like the previous result is immediate from the fact that regions can only switch from inactive to active as t_1 increases. It follows that the sets $\{t_1 \mid \rho(t_1) \leq r^k\}$ and $\{t_1 \mid \rho(t_1) \geq r^k\}$ are convex. I^k is the intersection of these sets so it is convex.

Finally, each component of the partition contains its supremum. Define $\tau^k = \sup(I^k)$, $k = 1, \ldots, M$. We want to show $\tau^k \in I^k$. Each component of the partition is bounded above by 1, so $\tau^k \leq 1$ for all k. Obviously $\tau^M = 1 \in I^M$, so we are done if $M = 1$ or $M > 1$ and $k = M$. The remaining case is $M > 1$ and $k < M$. Since $\rho(\cdot)$ is nondecreasing, we have $\rho(\tau^k) \geq r^k$, so all we need to establish is $\rho(\tau^k) \leq r^k$. To do this, we need to show that any region $j \geq 2$ that is inactive on I^k is also inactive at τ^k. So, suppose region j is inactive on I^k. From the elementary properties of a supremum, there must be a sequence of tax rates $\{t_1^{k,h}\}_{h=1}^{\infty}$ that lies entirely in I^k such that $\lim_{h\to\infty} t_1^{k,h} = \tau^k$. By assumption $S_j(t_1^{k,h}) = 0$, so $\lim_{h\to\infty} S_j(t_1^{k,h}) = 0$. We also know S_j is continuous at τ^k, since $t_{-1} \neq (1, \ldots, 1)$. It now follows that $S_j(\tau^k) = 0$, which says that region j is inactive at τ^k.

Putting these results together, we conclude that if $M \geq 2$ then the partition has the form

$$I^1 = [0, \tau^1],\ I^2 = (\tau^1, \tau^2],\ I^3 = (\tau^2, \tau^3],\ \ldots\ , I^M = (\tau^{M-1}, 1],$$

where $0 = \tau^0 \leq \tau^1 < \tau^2 < \cdots < \tau^{M-1} < \tau^M = 1$.

Step (iii). Now fix any value of k such that region 1 is active on (τ^{k-1}, τ^k). We want to show that S_1 is concave there.

Suppose $r^k = 0$, so region 1 is the only active region on the interval. It follows that $S_1(\cdot) = \bar{s}$, so we are done.

Suppose $r^k > 0$. It is convenient to renumber the active regions consecutively, so for regions $i = 1, \ldots, r^k + 1$ we have $S_i(\cdot) > 0$ on (τ^{k-1}, τ^k). For notational convenience, define

$$R \equiv r^k + 1.$$

From the definition of equilibrium, the functions $(S_1, \ldots, S_R)$ evaluated at t_1 solve

$$\begin{aligned}
(1 - t_1) f_1'(s_1) &= (1 - t_2) f_2'(s_2) \\
(1 - t_1) f_1'(s_1) &= (1 - t_3) f_3'(s_3) \\
&\ \vdots \\
(1 - t_1) f_1'(s_1) &= (1 - t_R) f_R'(s_R) \\
s_1 + \cdots + s_R &= \bar{s}.
\end{aligned}$$

We now apply the implicit function theorem. Each S_i is C^2 on (τ^{k-1}, τ^k) since each f_i is differentiable at least three times at the solution (this uses $s_i > 0$) and each equality in the system is smooth in t_1. Denoting

$$S_i' = \frac{\partial S_i}{\partial t_1}, \quad i = 1, \ldots, R.,$$

we have

$$\Gamma \begin{bmatrix} S_1' \\ S_2' \\ \vdots \\ S_{R-1}' \\ S_R' \end{bmatrix} = \begin{bmatrix} f_1' \\ f_1' \\ \vdots \\ f_1' \\ 0 \end{bmatrix},$$

where

$$\Gamma = \begin{bmatrix} (1-t_1) f_1'' & -(1-t_2) f_2'' & 0 & \dots & 0 \\ (1-t_1) f_1'' & 0 & -(1-t_3) f_3'' & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (1-t_1) f_1'' & 0 & 0 & \dots & -(1-t_R) f_R'' \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix}.$$

Differentiating again gives

$$\Gamma \begin{bmatrix} S_1'' \\ S_2'' \\ \vdots \\ S_{R-1}'' \\ S_R'' \end{bmatrix} = \begin{bmatrix} (1-t_2) f_2'''(S_2')^2 - (1-t_1) f_1'''(S_1')^2 + 2 f_1'' S_1' \\ (1-t_3) f_3'''(S_3')^2 - (1-t_1) f_1'''(S_1')^2 + 2 f_1'' S_1' \\ \vdots \\ (1-t_R) f_R'''(S_R')^2 - (1-t_1) f_1'''(S_1')^2 + 2 f_1'' S_1' \\ 0 \end{bmatrix}.$$

We need the formula for S_1' and the sign of S_1''. To use Cramer's rule, we need the determinant of a matrix of the form

$$A = \begin{bmatrix} a_1 & b_1 & 0 & \dots & 0 \\ a_2 & 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{R-1} & 0 & 0 & \dots & b_{R-1} \\ a_R & 1 & 1 & \dots & 1 \end{bmatrix}.$$

One can show by induction that[19]

$$\det(A) = (-1)^{R-1} \sum_{i=1}^{R} a_i \prod_{j \neq i}^{R} b_j, \quad b_R \equiv -1.$$

Define

$$b_j = -(1-t_{j+1}) f_{j+1}'', \quad j = 1, \dots, R-1.$$

[19] I am grateful to Professor Wilhelm Neuefeind for an elegant proof, which is available on request.

Then

$$S_1' = \frac{\begin{vmatrix} f_1' & b_1 & 0 & \dots & 0 \\ f_1' & 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_1' & 0 & 0 & \dots & b_{R-1} \\ 0 & 1 & 1 & \dots & 1 \end{vmatrix}}{\begin{vmatrix} (1-t_1)\,f_1'' & b_1 & 0 & \dots & 0 \\ (1-t_1)\,f_1'' & 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (1-t_1)\,f_1'' & 0 & 0 & \dots & b_{R-1} \\ 1 & 1 & 1 & \dots & 1 \end{vmatrix}}.$$

$$= \frac{f_1' \sum_{i=1}^{R-1} \prod_{j \neq i}^{R} b_j}{(1-t_1)\,f_1'' \sum_{i=1}^{R-1} \prod_{j \neq i}^{R} b_j + \prod_{j=1}^{R-1} b_j}$$

$$= \frac{f_1'}{(1-t_1)\,f_1'' + \left[\sum_{i=2}^{R} \frac{1}{(1-t_i)\,f_i''} \right]^{-1}}.$$

We use this formula below. Clearly $S_1' < 0$ since the numerator is positive and denominator is negative.

Define

$$c_i = (1-t_{i+1})\,f_{i+1}'''(S_{i+1}')^2 - (1-t_1)\,f_1'''(S_1')^2 + 2\,f_1'' S_1', \qquad i = 1, \dots, R-1.$$

Then

$$S_1'' = \frac{\begin{vmatrix} c_1 & b_1 & 0 & \dots & 0 \\ c_2 & 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{R-1} & 0 & 0 & \dots & b_{R-1} \\ 0 & 1 & 1 & \dots & 1 \end{vmatrix}}{\begin{vmatrix} (1-t_1)\,f_1'' & b_1 & 0 & \dots & 0 \\ (1-t_1)\,f_1'' & 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (1-t_1)\,f_1'' & 0 & 0 & \dots & b_{R-1} \\ 1 & 1 & 1 & \dots & 1 \end{vmatrix}}$$

$$= \frac{\sum_{i=1}^{R-1} c_i \prod_{j \neq i}^{R} b_j}{(1 - t_1) f_1'' \sum_{i=1}^{R-1} \prod_{j \neq i}^{R} b_j + \prod_{j=1}^{R-1} b_j.}$$

The denominator is positive and, apart from c_i, the numerator is negative, so $S_1'' < 0$ is assured if all of the c_i are positive. From the definition of c_i and the assumption $f_i''' > 0$, a sufficient condition for each c_i to be positive is $2f_1''S_1' - (1 - t_1)f_1'''(S_1')^2 > 0$. Dividing through by $S_1' < 0$ gives $2f_1'' - (1 - t_1) f_1'''S_1' < 0$. Using the formula for S_1', this holds if and only if

$$2f_1'' - (1 - t_1) f_1'''\left(\frac{f_1'}{(1 - t_1) f_1'' + \left[\sum_{i=2}^{R} \frac{1}{(1 - t_i) f_i''}\right]^{-1}}\right) < 0.$$

The denominator is negative, so multiplying through gives the condition

$$2f_1''(1 - t_1) f_1'' + 2f_1'' \left[\sum_{i=2}^{R} \frac{1}{(1 - t_i) f_i''}\right]^{-1} - (1 - t_1) f_1''' f_1' > 0.$$

The middle term is positive, so eliminating it and then dividing through by $(1 - t_1)$ gives, as a sufficient condition for $c_i > 0$, the condition $2(f_1'')^2 - f_1'''f_1' \geq 0$. We assume this in Assumpetion 9(*iii*). We conclude that S_1 is concave on (τ^{k-1}, τ^k).

Step (iv). Now fix two adjacent intervals, say I^k and I^{k+1}. We want to show that if region 1 is active on the union, (τ^{k-1}, τ^{k+1}), then S_1 is concave there. Since S_1 is concave on each interval separately, it is clear that all we need to show is that S_1 "becomes steeper" at the point where they join. Formally, we show that the left derivative of S_1 at τ^k is less (in absolute terms) than the right derivative.

We know that $S_1(\cdot)$ is continuous on $[0, 1]$ and concave on (τ^{k-1}, τ^k) so it is concave on $[\tau^{k-1}, \tau^k]$. From Rockafellar (1970), Theorem 23.1, the left derivative of S_1, written as $S_1'^{-}$, exists at τ^k. Furthermore, from Theorem 24.1, it equals the limit of the derivatives from below, $S_1'(t_1)$. We have an expression for these derivatives from our previous results.[20] Define $R^k \equiv r^k + 1$ (so R^k is the same as R above, but now we need the more general notation). Then

[20]Technically, Theorem 24.1 applies to closed and proper concave functions defined on the entire real line. We are actually taking the left derivative at τ^k of the function that agrees with S_1 on $[\tau^{k-1}, \tau^k]$ and is $-\infty$ on the remainder of $\Re$. This is closed and proper.

$$S_1'^{-}(\tau^k) = \lim_{t_1\uparrow\tau^k} S_1'(t_1) = \lim_{t_1\uparrow\tau^k} \frac{f_1'[S_1(t_1)]}{(1-t_1)\,f_1''[S_1(t_1)] + \left[\sum_{i=2}^{R^k} \frac{1}{(1-t_i)\,f_i''[S_i(t_1)]}\right]^{-1}}.$$

All terms on the right-hand side are continuous at τ^k since $S_i(\tau^k) > 0$, so

$$S_1'^{-}(\tau^k) = \frac{f_1'[S_1(\tau^k)]}{(1-\tau^k)\,f_1''[S_1(\tau^k)] + \left[\sum_{i=2}^{R^k} \frac{1}{(1-t_i)\,f_i''[S_i(\tau^k)]}\right]^{-1}}.$$

For the right derivative, note that the number of active regions is now $R^{k+1} = r^{k+1} + 1$. We have[21]

$$S_1'^{+}(\tau^k) = \lim_{t_1\downarrow\tau^k} S_1'(t_1) = \frac{f_1'[S_1(t_1)]}{(1-t_1)\,f_1''[S_1(t_1)] + \left[\sum_{i=2}^{R^{k+1}} \frac{1}{(1-t_i)\,f_i''[S_i(t_1)]}\right]^{-1}}.$$

Evaluating this limit requires us to evaluate

$$\lim_{t_1\downarrow\tau^k} \sum_{i=2}^{R^{k+1}} \frac{1}{(1-t_i)\,f_i''[S_i(t_1)]}.$$

This requires an extra step, since $S_i(\tau^k) = 0$ for all of the regions that are inactive on I^k but active on I^{k+1}. These are the regions numbered $R^k + 1$ through R^{k+1}. The limits for the corresponding terms exist by Assumption 9(*iv*). Therefore

$$S_1'^{+}(\tau^k) = \frac{f_1'[S_1(\tau^k)]}{(1-\tau^k)\,f_1''[S_1(\tau^k)] + \left[\sum_{i=2}^{R^k} \frac{1}{(1-t_i)\,f_i''[S_i(\tau^k)]} + \lim_{t_1\downarrow\tau^k} \sum_{i=R^k+1}^{R^{k+1}} \frac{1}{(1-t_i)\,f_i''[S_i(t_1)]}\right]^{-1}}.$$

It now follows that $|S_1'^{-}(\tau^k)| < |S_1'^{+}(\tau^k)|$ if and only if

$$\lim_{t_1\downarrow\tau^k} \sum_{i=R^k+1}^{R^{k+1}} \frac{1}{(1-t_i)\,f_i''[S_i(t_1)]} < 0.$$

This also follows from Assumption 9(*iv*).

[21]Again, to use Theorem 24.1, we are taking the right derivative at τ^k of the function that agrees with S_1 on $[\tau^k, \tau^{k+1}]$ and is $-\infty$ on the remainder of $\Re$.

We now know that if region 1 is active on (τ^{k-1}, τ^{k+1}) then $S_1(\cdot)$ is concave there. One can easily extend this result to show that if region 1 is active on any interval $(0, b)$, $0 < b \leq 1$, then it is concave there. We now use this result to complete the proof.

If region 1 is active on all of $[0, 1]$ then it is concave on $(0, 1)$ and, using continuity, concave on $[0, 1]$. We conclude that $u_1(\cdot)$ is quasiconcave on $[0, 1]$, by Theorem 1. Now suppose there is some tax rate in $[0, 1]$ at which region 1 is inactive. By continuity there is a smallest such tax rate, say t_1^*. If $t_1^* = 1$ then region 1 is active on $[0, 1)$. The previous argument gives the result. If $t_1^* \in (0, 1)$, then we have $S_1(\cdot)$ concave on $[0, t_1^*]$ and zero on $[t_1^*, 1]$. We have $u_1(\cdot)$ quasiconcave on $[0, t_1^*]$ by Theorem 1. $S_1(\cdot) = 0$ on $[t_1^*, 1]$ implies $c_1 = z_1 = 0$, so $u_1(\cdot) = U_1(0, 0)$ on $[t_1^*, 1]$. That is to say, the payoff function is flat on $[t_1^*, 1]$. The flat extension of a quasiconcave function is quasiconcave, so $u_1(\cdot)$ is quasiconcave on $[0, 1]$. Finally, if $t_1^* = 0$ then $S_1 = 0$ on all of $[0, 1]$ and the payoff function is flat and thus quasiconcave on $[0, 1]$. ■

Proof of Theorem 7: It is straightforward to verify that Assumptions 7–9 hold for each technology. ■

Proof of Theorem 8: Available on request. ■

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