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Order restricted preferences and majority rule

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Abstract. This paper develops the social choice foundations of a restriction that, in different guises, is utilized in a number of economic models; illuminates the key features of these models; and provides a specific class of applications. Order Restriction (on triples) is strictly weaker than Single Peakedness (or Single Cavedness) but strictly stronger than Sen's Value Restriction. It therefore guarantees quasi-transitivity of majority rule. This condition is most useful in models where there is a natural ordering of the individuals, not the alternatives. We show for a class of applications that *1)* it may be imposed through conditions with meaningful economic interpretations; *2)* imposing Single Peakedness weakens the usefulness of the models; *3)* directly establishing Value Restriction is difficult and has not been done.

1. Introduction

We define and analyze a condition on preferences, called Order Restriction, that prevents voting cycles under majority rule. In addition, we provide a class of applications and show that this condition holds in a variety of models.

More precisely, we define an exclusion restriction (Kramer [5]) that guarantees that the majority rule binary relation is quasitransitive, and so there is a best element (Condorcet winner) from any finite set of alternatives. Order Restriction (on triples) is strictly weaker than Single Peakedness (or Single Cavedness) but strictly stronger than Sen's Value Restriction (Sen [13]). Since the condition is stated in a

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fundamentally different way than is Value Restriction, establishing the precise relationship with it requires some care.[1]

In different guises, this restriction holds in papers by Hamada [3], Roberts [8], Grandmont [2], Hettich [4], Epple and Romer [1], Slesnick [15], and a number of others. Fundamentally, in these models there is a natural ordering of the individuals, not the alternatives, and relative to this ordering the preferences of individuals over any pair of alternatives obey a simple non-reswitching rule. This ordering may result from conditions like Roberts' Hierarchical Adherence (Roberts [8]) or others with meaningful economic interpretations. Verifying that these conditions produce Order Restricted preferences is straightforward. By comparison, Single Peakedness is unnecessarily restrictive and Value Restriction is very difficult to demonstrate. The fact that preferences are Value Restricted in these models has not been recognized, making their collective choice properties somewhat mysterious.

The class of applications results from synthesizing the models in Roberts [8] and Epple and Romer [1]. In these applications, individuals vote over tax and transfer proposals, preferences over the alternatives need not be Single Peaked nor Single Caved, and imposing these conditions is undesirably restrictive. Furthermore, Roberts and Epple-Romer testify to the difficulty of establishing Value Restriction in their models.[2]

The final section summarizes these results.

2. Collective choice theorems for order restricted preferences

We begin with an informal description of Order Restriction. The formal analysis follows, starting with a review of relevant definitions and results. We then proceed to the collective choice theorems. Proofs for all of the theorems are contained in this section.

2.1 An illustration

Suppose individuals have preferences over a set of alternatives S from which a collective choice is to be made. Exclude all individuals indifferent among all elements of S. Preferences are *Order Restricted on S* if there is a renumbering (or permutation) of the remaining individuals so that for each distinct pair of alternatives (say x and y), all of those who strictly prefer x to y are numbered lower than all of those who are indifferent between the two, and these are numbered lower than all those who strictly prefer y to x.

Equivalently, if we partition these individuals into three groups according to their preferences, we require the permutation:

[1] Sen and Pattanaik [14] thoroughly explore exclusion restrictions and develop all those necessary for explaining the absence of voting cycles. Any new condition yielding this result must be a reformulation of one or more of theirs. It is not our goal, however, to explain the absence of voting cycles. The goal is to establish the social choice theoretic foundations of a condition that in different guises has existed for some time in the literature and to isolate the key common feature of these applications.

[2] Roberts conjectures that preferences are not Value Restricted in his model. Romer [9] conjectures that they are, in his model.

(A) to group together all people who strictly prefer x to y, all who are indifferent, and all who strictly prefer y to x; and

(B) to place these groups in the order of strict preference, indifference, and strict (reverse) preference.

Any of these groups may be null, in which case requirement (B) is automatically met.

Example. Consider the following preference family on $S=\{x,y,z\}$:

$$\left\{\begin{matrix} (1) & (2) & (3) \\ x, & x, & y, \\ y & yz & z \\ z & & x \end{matrix}\right\}.$$

In this arrangement, $\{x,y\}$ and $\{x,z\}$ satisfy condition (A) but $\{y,z\}$ does not. Suppose we renumber the individuals by mapping (1) to (2′), (2) to (1′), and (3) to (3′). Then we obtain the second arrangement, where conditions (A) and (B) are met:

$$\left\{\begin{matrix} (1') & (2') & (3') \\ x, & x, & y, \\ yz & y & z \\ & z & x \end{matrix}\right\}.$$

The preference family is therefore Order Restricted on S.

2.2 *Preliminaries*

Following standard usage, suppose X is a set of alternatives (finite or infinite) containing at least three distinct elements. For $S\subseteq X$, a *binary relation on S* is a subset of S^2.

Preferences for the alternatives in X are represented by the binary relation R. Strict preference, P, is defined in terms of R by $\{(x,y)|xRy\,\&\,\neg yRx\}$. R is *complete* if $(\forall x,y\in X)(xRy \vee yRx)$, *transitive* if $(\forall x,y,z\in X)(xRy\,\&\,yRz\rightarrow xRz)$, and *quasi-transitive* if P is transitive. Quasi-transitivity is weaker than transitivity since it allows intransitive indifference.

A *triple*, $T\subseteq X$, contains three distinct alternatives. Given a binary relation R on X, the *restriction of R to T* is the binary relation on T defined by $R^T=R\cap T^2$. It is all members of R with both alternatives in T.

If M is an arbitrary set of integers and we have a preference relation associated with each $i\in M$, we will denote the entire *family of preference relations* by $\{R_i\}_{i\in M}$. This concept will be more useful to us than the traditional *preference profile*, which is an ordered collection of preference relations.

We suppose there is a finite set of individuals, N, with $\#N=n\geqq 3$. Each individual i has preferences R_i on X. Individual preferences are complete and transitive. We therefore have a fixed family of individual preference relations, $\{R_i\}_{i\in N}$.

The following notation is useful. N_S denotes the set of individuals concerned for S (individuals not indifferent among all the alternatives) and $C(R,S)$ denotes the set of best elements in S (alternatives that tie or defeat all others according to R). Finally, R_M denotes the majority rule binary relation.

Definition 1. Order Restriction on $T(OR_T)$.

If A and B are sets of integers, let $A > B$ mean that every element of A is greater than every element of B. Preferences are Order Restricted on T if there is a permutation $J: N_T \rightarrow N_T$ such that for all distinct x, y in T,

$$\{J(i)|xP_iy, i \in N_T\} > \{J(i)|xI_iy, i \in N_T\} > \{J(i)|yP_ix, i \in N_T\} \tag{1}$$

or

$$\{J(i)|xP_iy, i \in N_T\} < \{J(i)|xI_iy, i \in N_T\} < \{J(i)|yP_ix, i \in N_T\} \ . \tag{2}$$

If we replace every instance of "T" above with "S" we obtain the natural extension of Order Restriction to sets of alternatives larger than triples. It follows immediately that Order Restriction on S implies Order Restriction on every triple in S, and one may show that the converse is false, so OR_S is a strictly stronger requirement.[3]

Definition 2. Value Restriction on $T(VR_T)$.

Preferences are Value Restricted on T if there is a labeling of the elements of T as x, y, z such that

$$VR_T[1])(\forall i \in N_T)((xP_iy) \vee (xP_iz))$$

or

$$VR_T[2])(\forall i \in N_T)((yP_ix) \vee (zP_ix))$$

or

$$VR_T[3])(\forall i \in N_T)(((xP_iy)\&(xP_iz)) \vee (yP_ix)\&(zP_ix))) \ .$$

$VR_T[1]$ says that x is never on the bottom, or "not worst." $VR_T[2]$ says that x is "not best," and $VR_T[3]$ says that x is "not medium." Sen [13] establishes that $VR_T[1]$ is equivalent to Single Peakedness on T and $VR_T[2]$ is equivalent to Single Cavedness on T.[4]

The terms "voting cycle" and "paradox of voting " refer to situations where for some S, $C(R_M, S) = \emptyset$. It is important to know when this cannot happen, that is to say, when for every S, $C(R_M, S)$ is not empty.[5] The following result is along these lines.

Theorem 1 (Sen). *(a) If preferences satisfy VR_T on every triple $T \subseteq X$, then R_M is quasitransitive on X.*

(b) If R_M is quasitransitive on X and S is any finite and nonnull subset of X, then $C(R_M, S) \neq \emptyset$.

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[3] The family

$$\begin{Bmatrix} a, & b, & a, & d \\ b & a & d & a \\ c & d & b & b \\ d & c & c & c \end{Bmatrix}$$

satisfies OR_T for all $T \in S = \{a, b, c, d\}$ but not OR_S.

[4] By convention, the statement that preferences satisfy $VR_T[1]$ is an abbreviation for the statement that there exists a labeling of the elements of T as x, y, z such that condition $VR_T[1]$ holds (and similarly for the other two components).

[5] See Kelly [5] and Moulin [7] for a discussion of the importance of having a best alternative, or "Condorcet winner," under majority rule.

Proof. See Sen [12] and [13]. (a) is Theorem 10*1 of Sen [13] and Theorem VIII of Sen [12] (the latter offers a brief and self-contained proof). (b) is Lemma 1*k of Sen [13].

2.3 Theorems for order restricted preferences

We present the key results of this section, interpret them, and then offer proofs.

Theorem 2. *Given* $\{R_i\}_{i \in N}$ *and* T *a triple in* X,

(a) *If preferences satisfy* OR_T *then they satisfy* VR_T.

(b) *The converse of (a) is false.*

Corollary 2.1. *Given* $\{R_i\}_{i \in N}$, *if preferences satisfy* OR_T *on every triple* $T \subseteq X$, *then for any finite and nonnull* $S \subseteq X$, $C(R_M, S) \neq \emptyset$.

Theorem 3. *Given* $\{R_i\}_{i \in N}$ *and* T *a triple in* X,

(a) *If preferences satisfy* $VR_T[1]$ *or* $VR_T[2]$ *then they satisfy* OR_T.

(b) *The converse of (a) is false.*

Theorem 4. *Given* $\{R_i\}_{i \in N}$ *and* T *a triple in* X, *if all individuals have strict preferences for every pair, then preferences satisfy* $VR_T[1]$ *or* $VR_T[2]$ *if and only if they satisfy* OR_T.

Theorem 2 states that Order Restriction is strictly stronger than Value Restriction. This result may be somewhat surprising. When indifference can occur, the different alternatives quickly "pick up values," making it very likely that all of the alternatives will occupy all of the values. This is precisely what Value Restriction forbids. The Corollary then follows immediately from Theorem 2.

Theorem 3 states that on triples Order Restriction is strictly weaker than Single Peakedness or Single Cavedness. We may therefore attempt to apply it in models where these conditions need not hold.[6]

Theorem 4 states that Order Restriction is equivalent to Single Peakedness and Single Cavedness when individual indifference is not allowed.

Proof of Theorem 2(a)[7]. We establish the contrapositive. Suppose $\{R_i\}_{i \in N}$ fails VR_T. Then there must be three concerned individuals with preferences that may be written:

$$xR_i yR_i z\ ,\quad yR_j zR_j x\ ,\quad zR_k xR_k y\ .$$

Furthermore, we must have $xP_i z, yP_j x$, and $zP_k y$, otherwise one of the individuals is unconcerned.

We now consider the three distinct permutations of i, j, k. First, if

$$J(i) < J(j) < J(k)$$

[6] There is also the familiar concept of Single Peakedness on S, where S contains more than three elements. One can show that Single Peakedness on S and Order Restriction on S are independent conditions. However, Single Peakedness on S implies that Single Peakedness holds on every triple in S, which implies that Order Restriction holds on every triple in S (but not conversely), so Single Peakedness on S is strictly stronger than Order Restriction on T for every $T \in S$.

[7] I am grateful to a referee for suggesting the following approach, which significantly shortens the original proof.

then we have xR_iy, yP_jx, and xR_ky, so the conditions (1) and (2) fail. If the permutation is

$$J(j) < J(i) < J(k)$$

then we have zR_jx, xP_iz, and zR_jx and again they fail. Finally,

$$J(j) < J(k) < J(i)$$

produces yR_jz, zP_ky, and yR_iz.

For every distinct permutation there is a pair of alternatives on which (1) and (2) fail, and so OR_T fails.

Proof of Theorem 2(b). Consider the following preference family:

$$\begin{Bmatrix} (1) & (2) & (3) & (4) \\ a, & a, & b, & c \\ b & c & c & b \\ c & b & a & a \end{Bmatrix} .$$

With $x=a$, $VR_T[3]$ holds. But preferences are not Order Restricted. To show this, we must establish that for every permutation, there exists a pair of alternatives (perhaps depending on the permutation) such that (1) and (2) fail. First, notice that $\{b, c\}$ fails conditions (1) and (2) with the given (identity) permutation. Second, since a is ranked first for (1) and (2) and last for (3) and (4), individuals (1) and (2) must remain grouped together, as must (3) and (4), under any other permutation.

This leaves few possibilities. We can transpose (1) and (2) or (3) and (4), separately or together; we can move the groups, yielding (3) (4) (1) (2); and we can then transpose the pairs again, separately or together. In all cases, conditions (1) and (2) fail for some pair (in fact they always fail on $\{b, c\}$). Therefore OR_T fails.

Proof of Theorem 3(a). Suppose $\{R_i\}_{i \in N_T}$ satisfies $VR_T[1]$. Then there is a labeling of the elements of T as $\{x, y, z\}$ such that

$$xP_iy \vee xP_iz, i \in N_T .$$

Using the definitions we can without loss of generality relabel the alternatives so $T=\{x, y, z\}$ and the above condition holds.

We now claim:

$$\{R_i^T\}_{i \in N_T} \subseteq \begin{Bmatrix} y, & xy, & x, & x, & x & xz, & z, & xyz \\ x & z & y & yz & z & y & x & \\ z & & z & & y & & y & \end{Bmatrix} = Q ,$$

where $\{R_i^T\}_{i \in N_T}$ is the preference family restricted to T. Suppose not. Then there exists R_{i*}^T with x strictly preferred to y or x strictly preferred to z and which does not belong to Q. There are only thirteen transitive and complete binary relations on the triple $\{x, y, z\}$ and the family above contains eight of them.[8] Therefore R_{i*}^T must be one of the five remaining:

$$\begin{Bmatrix} y, & yz, & z, & z, & y \\ z & x & y & yx & xz \\ x & & x & & \end{Bmatrix} .$$

[8] See Sen [13, p 175].

All of these have y preferred or indifferent to x and z preferred or indifferent to x, a contradiction.

Since $\{R_i^T\}_{i\in N_T} \subseteq Q$ and Q satisfies OR_T with the given arrangement of the individuals, we immediately conclude $\{R_i\}_{i\in N_T}$ satisfies OR_T.

A similar argument holds if preferences satisfy $VR_T[2]$.

Proof of Theorem 3(b). Consider the following preference family on T:

$$\left\{\begin{matrix} a, & a, & bc, & c \\ b & bc & a & b \\ c & & & a \end{matrix}\right\}.$$

This clearly satisfies OR_T.[9] On the other hand, it fails $VR_T[1]$ and $VR_T[2]$: every alternative is best for someone and worst for someone.

Proof of Theorem 4. We show that if $\{R_i^T\}_{i\in N_T}$ fails $VR_T[1]$ and $VR_T[2]$ then it fails OR_T. The converse follows from Theorem 3(a).

Let Ω denote the set of preference families that contain only strict rankings and which fail $VR_T[1]$ and $VR_T[2]$. It is sufficient to show that every member of Ω fails OR_T. There are only six strict rankings of a triple, and clearly every member of Ω must contain three or more rankings.

Consider the members of Ω containing exactly three rankings. Since all three alternatives must occupy the top and bottom positions, there are only two such families:

$$\left\{\begin{matrix} x, & y, & z \\ y & z & x \\ z & x & y \end{matrix}\right\}\cdot \quad \left\{\begin{matrix} x, & y, & z \\ z & x & y \\ y & z & x \end{matrix}\right\}.$$

Each of these fails OR_T.

Now consider the members of Ω containing exactly four rankings. There are $6!/(4!2!) = 15$ distinct families with four rankings. Six of these 15 contain all the rankings in one of the families above, and so belong to Ω and also fail OR_T. This leaves 9 cases to check. Of these, only three belong to Ω, and they fail OR_T (they are relabelings of the family in Theorem 2(b)).

Every preference family containing five or six rankings contains all the rankings in one of the families above, and so belongs to Ω and fails OR_T.

3. Order restricted preferences in economic models

Roberts' Hierarchical Adherence, Grandmont's Intermediate Preferences (in one dimension), and the assumptions on utility, tax, and social welfare functions in Epple-Romer [1], Hamada [3], Hettich [4], and Slesnik [15] all produce Order Restricted preference families. Fundamentally, these models belong to a class in which it is more natural to work with orderings of the individuals than with orderings of the alternatives. We illustrate this point with a model that synthesizes those of Roberts and Epple-Romer and provides a class of applications.

Individual i's preferences are represented by a differentiable utility function, $U^i(x^i, z^i)$, that is strictly increasing in both arguments and quasiconcave. Good x is

[9] It is also Value Restricted: with $x = a$, $VR_T[3]$ is satisfied.

the numeraire and the individual's endowments are x_0^i and z_0^i. The voting is over alternatives that offer a transfer of the numeraire, T_x (the same for everyone), and one of two taxes: a unit tax on z, t_z, or an "income" tax on x, t_x (explained more precisely below). The gross price of z under t_z is $p_G(t_z)$ and the net price is $p_N(t_z)$. We denote the two types of proposals as (t_x, T_x) and (t_z, T_x), respectively. Notice that we do not assume that government taxes and spending meet a budget constraint.

The individual maximizes

$$U^i(x^i, z^i)$$

subject to

$$x^i + (1-t_x)p_G(t_z)z^i = (x_0^i + T_x) + (1-t_x)p_N(t_z)z_0^i \ .$$

This formulation of the constraint proves to be appropriate for our purposes. Notice that the tax t_x changes the relative price of z in the same way whether it is bought or sold, while the tax t_z produces an equilibrium gross price at which z is bought and a net price at which it is sold.

The Lagrangian for the problem is:

$$\mathscr{L} = U^i(x^i, z^i) + \lambda[x^i + (1-t_x)p_G(t_z)z^i - (x_0^i + T_x) - (1-t_x)p_N(t_z)(z_0^i)] \ .$$

This yields indirect utility, V^i, and demand for z, z^{*i}, which are functions of the proposal parameters. We will use the following expressions, derived from the envelope theorem:

$$-\frac{\partial V^i/\partial t_x}{\partial V^i/\partial T_x} = p_N z_0^i - p_G z^{*i} \ , \tag{3}$$

and

$$-\frac{\partial V^i/\partial t_z}{\partial V^i/\partial T_x} = p_G' z^{*i} - p_N' z_0^i \ . \tag{4}$$

Example 1 (Roberts)

In this model, z is leisure time and x is a composite consumption bundle. All individuals have the same endowment, 24 leisure hours, and earn w for each hour they work. The government taxes labor earnings and offers a transfer of numeraire. Individuals vote over proposals of the form (t_x, T_x).

We can obtain the Roberts model in this setting by making the assignments $z_0^i = 24$, $p_N = p_G = w$, and $x_0^i = 0$. The budget constraint becomes:

$$x^i = (1-t_x)(24 - z^i)w + T_x$$

Therefore t_x is a tax on labor income.[10] Finally, (3) yields:

$$-\frac{\partial V^i/\partial t_x}{\partial V^i/\partial T_x} = w(24 - z^{*i}) \ , \qquad -\frac{\partial V^j/\partial t_x}{\partial V^j/\partial T_x} = w(24 - z^{*j}) \ . \tag{5}$$

[10] Equivalently, it is a tax on the amount of numeraire demanded in excess of the transfer.

To obtain collective choice results, Roberts explores a restriction on the family of preferences. Preferences satisfy *Hierarchical Adherence* if there is a renumbering of the individuals so that regardless of the proposal, $j > i$ implies $z^{*j} \leqq z^{*i}$.[11] In other words, person i demands more leisure than does person j, under every proposal. This requirement is remarkable for its simplicity, plausibility, and usefulness.[12]

Preferences over alternative tax and transfer pairs need not be Single Peaked in this model, even if we also impose a budget constraint. This is an advantage because requiring Single Peakedness would weaken the applicability of the model. Thus, if it were required, one could not in general choose unemployment under income tax proposals (or no consumption of z under excise tax proposals).[13]

Roberts shows directly that voting cycles will not occur. Left this way, the result is mysterious, but since preferences are Order Restricted, they are Value Restricted. Since this structure exists, it is less surprising that his direct proof works. We now briefly illustrate how Hierachical Adherence leads to Order Restriction.

Fix a pair of proposals, say $u=(t_x^1, T_x^1)$ and $v=(t_x^2, T_x^2)$, and suppose without loss of generality, $T_x^2 > T_x^1$. Suppose further that some individual i is indifferent between u and v.[14] By Hierarchical Adherence,

$$j > i \rightarrow z^{*j} \leqq z^{*i} \leftrightarrow w(24 - z^{*j}) \geqq w(24 - z^{*i})$$

(these terms must be positive because there is no endowment of the numeraire commodity; otherwise we would have to explicitly assume that individuals cannot purchase more than 24 hours of leisure). By (5), we conclude that j's indifference curves are not flatter than i's through any point, including all points along i's indifference curve. As Fig. 1 shows, this implies that j must find u preferred or indifferent to v.

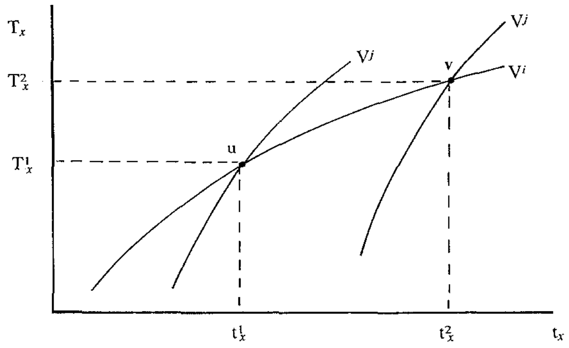

Fig. 1. Indifference curves for i and j. $u=(t_x^1, T_x^1)$; $v=(t_x^2, T_x^2)$; uI_iv; $j > i \rightarrow uR_jv$

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[11] Roberts works with labor income, $y^* \equiv (24 - z^*)w$, defines utility over x and y (so utility is a decreasing function of y), and assumes Hierarchical Adherence for y^*. The approaches are equivalent.

[12] It is in principle testable. However, while it is important to know the empirical circumstances under which people's preferences are likely to obey this condition, we suggest that its formal properties are interesting apart from these considerations.

[13] Roberts discusses some of these issues. For example, one can draw indifference curves for four individuals that satisfy Hierarchical Adherence but generate the non-Single Peaked preferences used in Theorem 3(b). With strict quasi-concavity and an aggregate budget constraint, Hierarchical Adherence does not remove the possibility of non-Single Peakedness in the tax rate, since aggregate labor supply may behave erratically.

[14] As long as one proposal is not unambigously better by offering a higher transfer and lower tax rate, we can assume such an individual exists.

If we order individuals according to z^* and set $x=v$ and $y=u$, all who prefer x to y occur first, then those who are indifferent, then those who prefer y to x. Order Restriction therefore holds.

Example 2 (Epple-Romer)

In this model z is housing and x is again a composite consumption bundle. Individuals have differing endowments of numeraire and no endowment of housing. The government places a unit tax on housing and offers a transfer of numeraire. Individuals vote over proposals of the form (t_z, T_x).

We can derive the Epple-Romer model with the assignments $z_0^i=0$ and $t_x=0$. This produces the budget constraint found in their paper,

$$x^i+p_G(t_z)z^i=x_0^i+T_x$$

and (4) yields:

$$-\frac{\partial V^i/\partial t_z}{\partial V^i/\partial T_x}=p_G' z^{*i}\ , \qquad -\frac{\partial V^j/\partial t_z}{\partial V^j/\partial T_x}=p_G' z^{*j} \tag{6}$$

(we suppose $p_G'>0$).

Epple and Romer do not assume Hierarchical Adherence. Rather, they suppose that all individuals have the same strictly quasi-concave utility function and that housing, z, is a normal good. These conditions are sufficient for Order Restriction; in fact, they imply Hierarchical Adherence.

To prove this, we must show that under all proposals, for all $j>i$, $z^{*j} \leqq z^{*i}$. Renumber the n individuals according to endowments, so $x_0^j \leqq x_0^i$ if and only if $j>i$, and suppose $j>i$. Since all utility functions are the same, we can write

$$z^{*i}=z^*(p_G(t_z), x_0^i+T_x)\ .$$

Fix an arbitrary pair of proposals, say (t_z^1, T_x^1) and (t_z^2, T_x^2). Since z is normal, we know

$$z^{*j}=z^*(p_G(t_z^1), x_0^j+T_x^1) \leqq z^*(p_G(t_z^1), x_0^i+T_x^1)=z^{*i}$$

and

$$z^{*j}=z^*(p_G(t_z^2), x_0^j+T_x^2) \leqq z^*(p_G(t_z^2), x_0^i+T_x^2)=z^{*i}\ .$$

Since the proposals were arbitrary, Hierarchical Adherence follows. The same argument as before, using (6) instead of (5), establishes Order Restriction.

We cannot synthesize all of the economic applications of Order Restricted preferences in one model. Therefore, we briefly describe the other papers mentioned earlier. In all of these, finding a suitable ordering of the individuals is more natural than finding one for the alternatives, and Rothstein [11] shows that Order Restriction holds.

In Hamada [3], individuals vote over the distribution of income, which is represented by a four component vector. He provides a direct and somewhat difficult proof that cycles will not occur. We show that Order Restriction holds with the ordering determined by the weight individuals place on the second component of the distribution vector.

Grandmont [2] develops a condition that in one dimension is strictly stronger than Order Restriction. Specifically, his condition implies that for any x, y, all individuals in $\{J(i)|xI_iy, i \in N_T\}$ must have the same preferences over the *entire* set of alternatives. By his fundamental Proposition, all of these individuals are tied to the same point on the real line, and this maps to a unique ranking of the alternatives. This requirement is not met, for example, if there are three alternatives and the distinct rankings in the preference family are $xIyPz$, $zPxIy$, and $xPyPz$. Yet, these preferences are Order Restricted. One implication is that Theorem 2(a) improves his Lemma 1, since the premises are strictly weaker (preferences like those above are not excluded) but the essential conclusion, that cycles do not result, still holds.[15]

Hettich [4] develops an index of horizontal inequity in tax reform. The index depends on a parameter that determines the progressivity of the tax schedule. Hettich asks, but does not answer, the question of whether cycles will occur if individuals vote for reforms according to the level of inequity given by the index. We show that if we order the individuals according to their taste for progressivity, then preferences are Order Restricted.

Finally, Slesnick [15] provides an application of Hierarchical Adherence with empirical implications. He develops and estimates a model of income redistribution in which he compares the amount of redistribution possible under three reallocation mechanisms and two political decision rules. In the case of optimal redistributional policy under majority rule, individuals vote over a single parameter corresponding to the minimum level of welfare the policy guarantees. Slesnick shows that in his model the ordering of the individuals by ex post welfare levels is the same under each alternative, so it satisfies a Hierarchical Adherence condition. Preferences are therefore Order Restricted. With this, he estimates equilibrium levels of inequality under the six regimes.

4. Conclusion

In summary, we present a restriction on preferences and a class of applications. The restriction is strictly weaker on triples than Single Peakedness or Single Cavedness but strictly stronger than Sen's Value Restriction. If it holds for any finite set of alternatives, or for every triple in any finite set, there is a Condorcet winner from this set. The condition is also somewhat easier to state than Value Restriction.

In terms of applications, it may be imposed in economic models in a number of ways, most notably through Hierarchical Adherence. Imposing Single Peakedness would weaken these models, while directly establishing Value Restriction is difficult. It is also a device for revealing that Value Restriction holds in a class of models where this fact has gone unrecognized.

Restrictions on preferences are a tricky subject. One may always find circumstances where a restriction is unlikely to hold as a matter of empirical fact. On the other hand, analyzing restrictions can deepen our understanding of our models. We show, at the very least, that a number of diverse models have a common structure that has gone unrecognized and develop pure theorems of social choice that apply when this structure holds.

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[15] Grandmont's Lemma 1 states that under his assumptions, preferences will be Single Peaked or Single Caved for every "non-trivial" triple. The preferences given in the text satisfy non-triviality, fail his other assumptions, and satisfy only $VR_T[3]$ as well as OR_T.

We also not that Grandmont establishes his collective choice results by identifying the majority rule binary relation with an element of the preference family. Order Restriction provides a similiar link between majority rule outcomes and the preferences of the individual who is median under the permutation. Rothstein [11] develops these results in detail.

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