Lecture 9

Outline

1. Optimal tax problem as minimizing total excess burden with the fixed utility measure
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3. Optimal tax problem as minimizing total excess burden with the variable utility measure
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1. Optimal tax problem as minimizing total excess burden with the fixed utility measure

   (a) Recall equation (1) from Lecture 8:

   $$\mathrm{CF}_t^{\mathtt{LD}}(\hat{t}) = E(q^o + \hat{t}, V_t) - E(q^o, V_t) - \hat{t}x^c(q^o + \hat{t}, V_t)$$

   (Note that $\hat{t}$ is a vector – we omit the transpose sign to keep the notation simple.)
   Let t^* denote the optimal tax vector and use this and initial income to define V_{t^*}.
   The problem is:

   $$\underset{\hat{t}_1,...,\hat{t}_n}{\text{Min}} \ \mathrm{CF}_{t^*}^{\mathtt{LD}}(\hat{t})$$
   $$\text{subject to: } \hat{t}x^c(q^o + \hat{t}, V_{t^*}) = R$$

   Using the constraint to simplify the objective function gives:

   $$\underset{\hat{t}_1,...,\hat{t}_n}{\text{Min}} \ E(q^o + \hat{t}, V_{t^*}) - E(q^o, V_{t^*}) - R$$
   $$\text{subject to: } \hat{t}x^c(q^o + \hat{t}, V_{t^*}) = R$$

After eliminating the constants, the Lagrangian is:

$$\mathcal{L} = E(q^o + \hat{t}, V_{t^*}) - \lambda[\hat{t}x^c(q^o + \hat{t}, V_{t^*}) - R]$$

The first order condition for the kth commodity is:

$$x_k^c(.) - \lambda \left[x_k^c(.) + \sum_{i=1}^{n} \hat{t}_i S_{ik}(.) \right] = 0, \quad k = 1, ..., n$$

Therefore:

$$\frac{1-\lambda}{\lambda} = \frac{\sum_{i=1}^{n} \hat{t}_i S_{ik}(.)}{x_k^c(.)}, \qquad k = 1, ..., n$$

This gives a set of equations that are equivalent to those obtained from maximizing utility subject to the revenue constraint. The loss minimization problem is consistent with the optimal tax problem, given this measure of total loss.

2. Marginal excess burden using the variable utility measure (graph only)

   (a) Recall:

   $$\text{TEB}^{\text{LD}}(t) \;=\; I^o - E[q^o, V(q^o + t, I^o)] - tx(q^o + t, I^o)$$

   (b) The derivative with t becomes a bit tricky. You have the derivative of the expenditure function with utility and then the derivative of indirect utility with price.

   Note that you cannot use duality to cancel the terms. While the first is the reciprocal of the marginal utility of income and the second is the marginal utility of income, they are evaluated at different price vectors.

   (c) Gronberg and Liu (2001) develop an expression for this measure of marginal excess burden.

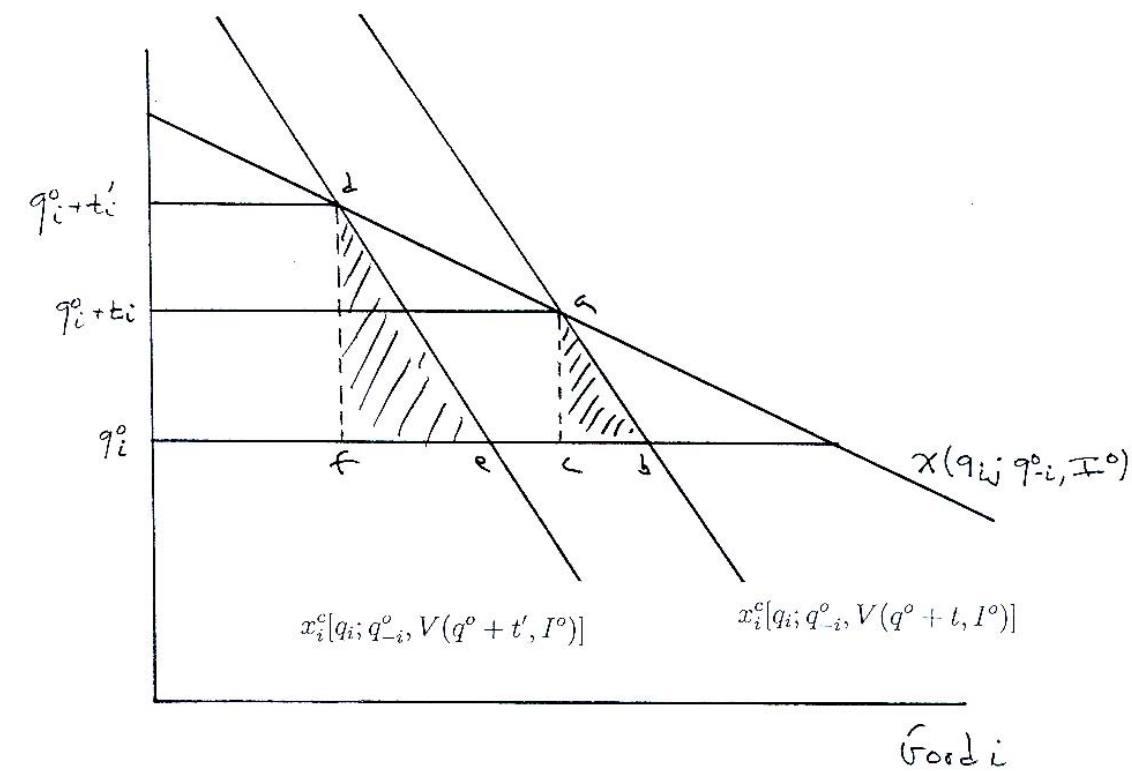
   Kay (1980) also does a little work with this.

   (d) A picture:

Figure 1

3. Optimal tax problem as minimizing total excess burden with the variable utility measure

   (a) Consider choosing t to minimize $\text{TEB}^{\text{LD}}(t)$ subject to a revenue constraint.



MEB ≈ def − abc

Figure 1

We have:

$$\begin{array}{l} \text{Min} \;\; I^o - E[q^o, V(q^o+t, I^o)] - tx(q^o+t, I^o) \\ t_1, ..., t_n \\ \text{subject to: } \; tx(q^o+t, I^o) = R \end{array}$$

As above, we can use the constraint to simplify the objective function:

$$\begin{array}{l} \text{Min} \;\; I^o - E[q^o, V(q^o+t, I^o)] - R \\ t_1, ..., t_n \\ \text{subject to: } \; tx(q^o+t, I^o) = R \end{array}$$

The objective function is *decreasing* in utility and we are *minimizing* the objective function. This is formally identical to the problem that gave us the Ramsey rule.

(b) Both Kay (1980) and Auerbach (1985) consider this.

4. Total excess burden one more time (general equilibrium or "$o\mathtt{R}$" measure)

   (a) We drew the picture for this at the start of Lecture 7. Now we formalize it.

   (b) There is some initial set of prices and income. The government imposes a tax vector t. This raises some revenue. The revenue is returned lump-sum at the post-tax prices. The individual achieves some level of utility at the new income and prices, but is not in general returned to the level of utility achieved at the initial state.

   The excess burden of the tax vector t is defined as the (negative of the) the equivalent variation of the transition from the initial state to this "lump-sum return" state.

   (c) Formally:

$$\begin{array}{ll} \text{state } o \text{ (no taxes)} : & \begin{array}{l} q^o \\ I^o \end{array} \\ & \\ \text{state } \mathtt{R} \text{ (lump-sum-return)} : & \begin{array}{rl} q^{\mathtt{R}} = & q^o + t \\ I^{\mathtt{R}} = & I^o + tx(q^o+t, I^o) \end{array} \end{array}$$

$$\text{Excess Burden} \;\equiv\; -EV^{o\mathtt{R}} = I^o - E[q^o, V(q^{\mathtt{R}}, I^{\mathtt{R}})]$$

Adding and subtracting the tax revenue gives:

$$\begin{aligned} -EV^{o\mathtt{R}} &= I^o + tx(q^o+t, I^o) - E[q^o, V(q^{\mathtt{R}}, I^{\mathtt{R}})] - tx(q^o+t, I^o) \\ &= I^{\mathtt{R}} - E[q^o, V(q^{\mathtt{R}}, I^{\mathtt{R}})] - tx(q^o+t, I^o) \end{aligned}$$

This gives the measure of total excess burden with all dependence on t explicit:

$$\text{TEB}^{o\mathtt{R}}(t) \;=\; I^{\mathtt{R}}(t) - E[q^o, V(q^{\mathtt{R}}(t), I^{\mathtt{R}}(t))] - tx(q^o+t, I^o)$$

(d) As before, we define a characterization function based on convenience. Define:

$$V_t \equiv V[q^o + t, I^o + tx(q^o + t, I^o)] = V(q^{\mathtt{R}}, I^{\mathtt{R}})$$

Now define:

$$\mathrm{CF}_t^{o\mathtt{R}}(\hat{t}) \;=\; E(q^o + \hat{t}, V_t) - E(q^o, V_t) - \hat{t}x(q^o + \hat{t}, I^o)$$

Note that this isn't *quite* the same as $\mathrm{CF}_t^{\mathtt{LD}}(\hat{t})$ because we keep Marshallian demand in the expression.

(e) To study triangles:

$$\mathrm{CF}_t^{o\mathtt{R}}(\hat{t})$$

$$= \sum_{i=0}^{n} \int_0^{\hat{t}_i} x_i^c(q_0^o + \hat{t}_0, ..., q_i^o + \tau, q_{i+1}^o, ..., q_n^o, V_t)\mathrm{d}\tau - \hat{t}x(q^o + \hat{t}, V_t)$$

$$= \sum_{i=0}^{n} \int_0^{\hat{t}_i} x_i^c(q_0^{\mathtt{R}}, ..., q_i^o + \tau, q_{i+1}^o, ..., q_n^o, V_t)\mathrm{d}\tau - \hat{t}x(q^o + \hat{t}, V_t)$$

Of course, this holds in particular when $\hat{t} = t$.

For a single price change:

$$\mathrm{CF}_t^{o\mathtt{R}}(\hat{t}) = \int_0^{\hat{t}_i} x_i^c(q_i^o + \tau; q_{-i}^o, V_t)\mathrm{d}\tau - \hat{t}_i x_i(q_i^o + \hat{t}_i; q_{-i}^o, I^o)$$

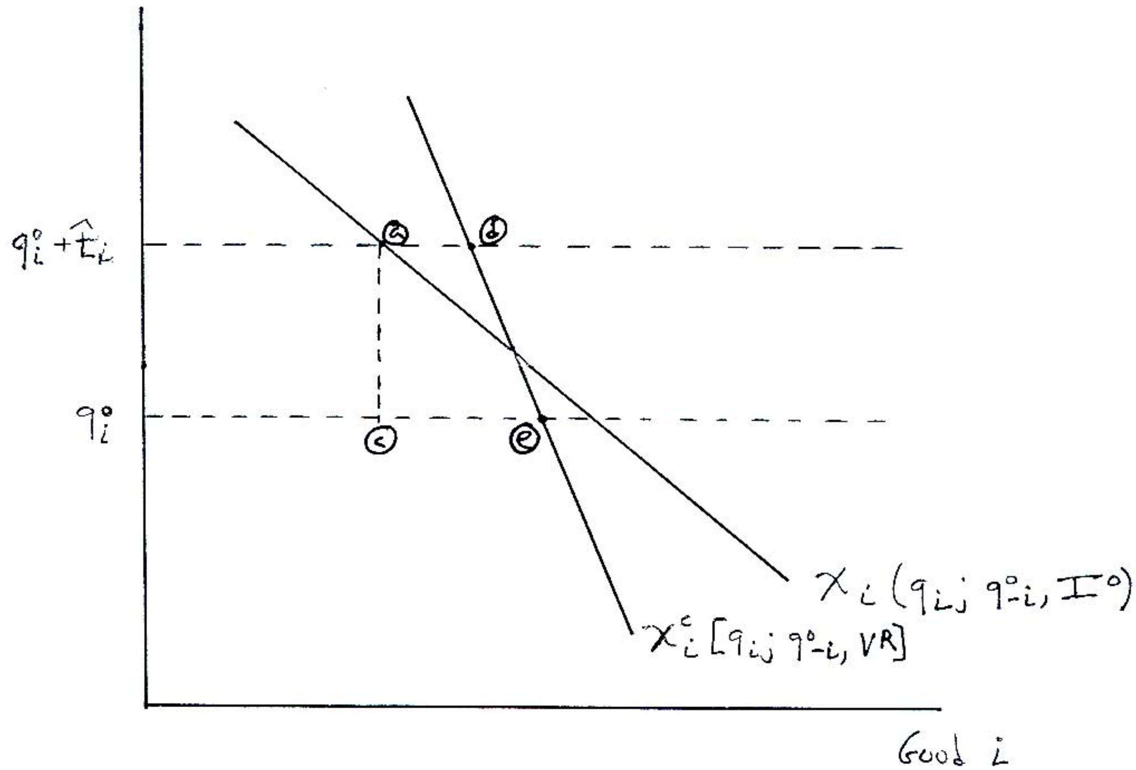
We draw the normal good case, so point "d" is to the right of point "a." Formally:

$$\begin{aligned} x_i^c(q_i^o + \hat{t}_i; q_{-i}^o, V_t) &\equiv x_i^c[q_i^o + \hat{t}_i; q_{-i}^o, V(q^o + \hat{t}, I^o + tx(.))] \\ &= x_i[q_i^o + \hat{t}_i; q_{-i}^o, I^o + tx(.)] \\ &> x_i(q_i^o + \hat{t}_i; q_{-i}^o, I^o) \end{aligned}$$

Figure 2

The area defined (although not the picture) is identical to that in Boadway and Bruce, Figure 7.6.

Figure 2



DWL = adec

5. Introduction to special cases

   (a) Certain restrictions on preferences simplify the analysis of excess burden and are used a lot in various literatures.

   Each of these special cases has implications for the shape of indifference curves and the relationship between standard and compensated demand. Two of the special cases have distinct implications for the "marginal utility of income."

   (b) Notation for commodities and prices:

$$x = (x_1, ..., x_n), \quad m$$

$$p = (p_1, ..., p_n), \quad p_m$$

   (c) Utility maximization:

$$\max_{x,m} \; U(x,m) \quad \text{subject to:} \quad px + p_m m = y$$

   Lagrangian:

$$\mathcal{L} = U(x,m) + \lambda(y - px - p_m m)$$

   Solution:

$$x_i(p, p_m, y), \quad i = 1, ..., n$$

$$m(p, p_m, y)$$

$$\lambda(p, p_m, y)$$

   Indirect utility function:

$$v(p, p_m, y) \equiv U[x_1(p, p_m, y), ..., x_n(p, p_m, y), m(p, p_m, y)] \tag{1}$$

   (d) Key properties:

      i. By the envelope theorem:

$$\frac{\partial v(p, p_m, y)}{\partial p_i} = -\lambda(p, p_m, y) x_i(p, p_m, y), \quad i = 1, ..., n \tag{2}$$

$$\frac{\partial v(p, p_m, y)}{\partial y} = \lambda(p, p_m, y) \tag{3}$$

      Therefore $\lambda(p, p_m, y)$ is the marginal utility of income.

      ii. Samuelson (1942) may have been the first to prove that *the marginal utility of income cannot be constant in all prices and incomes.* Any assumption that the marginal utility of income is "constant" must say precisely which variables are involved.

Formally, $\lambda(p, p_m, y)$ cannot be constant in all $(n+2)$ of its arguments. The proof is slick. The indirect utility function is homogeneous of degree zero (use (1) and the fact that $x_i(.)$ and $m(.)$ have this homogeneity). The marginal utility of income is therefore homogeneous of degree -1 (by Euler's theorem). Thus, doubling all prices and income must *halve* the marginal utility of income.

6. Quasi-linear preferences

   (a) Indifference Curves

   $$U(x, m) = u(x) + m$$

   In the 2-good case, (x_1, m) with x_1 on the horizontal axis, indifference curves are "vertically parallel." As long as we are at interior solutions, additional income has no effect on demand for x_1.

Figure 3

   (b) Compensated demand

   As long as we are at interior solutions, the demand curves $x_i(p, p_m, y)$ are independent of y. They can be written:

   $$x_i(p, p_m, y) = x_i(p, p_m)$$

   For each good i, standard demand and compensated demand overlap.

   (c) Slutsky equation

   The effect of a price change on standard demand comes entirely from the substitution effect. This is another way of understanding why standard and compensated demand overlap.

   (d) Excess burden

   As long as we are at interior solutions and good m is untaxed, excess burden can be computed exactly from the standard demand curves for each x_i since they are *also* the compensated demand curves.

   (e) Characterization using the marginal utility of income

   Samuelson (1942) showed that preferences are quasi-linear if and only if the marginal utility of income can be written as a function of p_m alone:

   $$\lambda(p, p_m, y) = \lambda(p_m) \tag{4}$$

   Silberberg (1978) has a nice demonstration that (4) implies that each $x_i(.)$ is independent of y. This isn't identical to what Samuelson showed, but

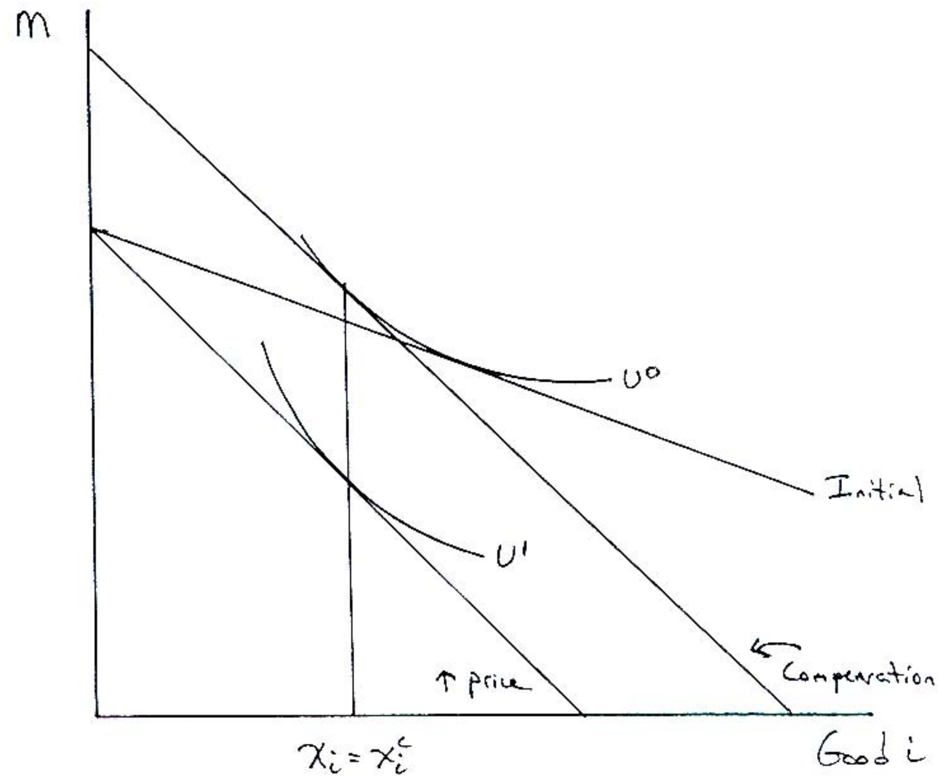


Figure 3

it takes you most of the way there. Using equations (2) and (3) we can derive, for all i:

$$-\left(\frac{\partial \lambda}{\partial y}x_i + \lambda\frac{\partial x_i}{\partial y}\right) = \frac{\partial^2 v(p, p_m, y)}{\partial p_i \partial y} = \frac{\partial^2 v(p, p_m, y)}{\partial y \partial p_i} = \frac{\partial \lambda}{\partial p_i}$$

The restriction in (4) implies that the right hand side is zero and the first term on the left hand side is zero. Therefore $\frac{\partial x_i}{\partial y} = 0$, all i.

(f) Example

A simple common example is:

$$U(x_1, m) = x_1^a + m$$

Maximizing subject to $p_1 x_1 + p_m m = y$ gives:

$$x_1(p_1, p_m, y) = \left(a\frac{p_m}{p_1}\right)^{\frac{1}{1-a}}$$

$$m(p_1, p_m, y) = \frac{y}{p_m} - \frac{p_1}{p_m}\left(a\frac{p_m}{p_1}\right)^{\frac{1}{1-a}}$$

$$\lambda(p_1, p_m, y) = \frac{1}{p_m}$$

7. Homothetic preferences (monetary value of a utility change)

(a) Indifference curves

Given any pair of goods, the marginal rate of substitution is constant along all points on any ray from the origin.

This implies that the ratio of chosen quantities will remain fixed as income varies.

This in turn implies that a $k\%$ increase in income causes a $k\%$ increase in demand for all goods.

Thus, the income elasticity of (standard) demand for all goods is 1. You can take this as the *definition* of homothetic preferences.

(b) Characterization using the marginal utility of income

Samuelson (1942) showed that preferences are homothetic if and only if the marginal utility of income can be written as a function of y alone:

$$\lambda(p, p_m, y) = \lambda(y) \tag{5}$$

(c) Money value of a utility change

Given homothetic preferences, there is a welfare interpretation to the area to the left of the standard demand curve.

In some applied contexts this measure is convenient. You can go directly from market equilibria using standard supply and demand curves to welfare conclusions.

While this may be *convenient*, it *does not* represent the equivalent variation of a price change. One must still use the compensated demand curves to represent that.

Indeed, if you take the philosophical position of Dupuit, areas to the left of the standard demand curve are always *superfluous*. Dupuit would say that welfare economics should not be based on the monetary values of interior states of well-being, it should be based on the concept of "willingness to pay." This is generally well-defined, it corresponds to intelligible thought experiments, it is linked to behavior, and is represented by areas to the left of compensated demand curves.

(d) For any preferences, homothetic or not, as long as indirect utility is differentiable (with a slight abuse of notation, the price p_m is the price p_{n+1}):

$v(p^1, p^1_m, y) - v(p^0, p^0_m, y)$

$= \sum_{i=1}^{n+1} \int_{p_i^0}^{p_i^1} \frac{\partial v(p_1^1, ..., p_{i-1}^1; s_i; p_{i+1}^0, ..., p_{n+1}^0, y)}{\partial s_i} \mathrm{d}s_i$

$$= - \sum_{i=1}^{n+1} \int_{p_i^0}^{p_i^1} x_i(p_1^1, ..., p_{i-1}^1; s_i; p_{i+1}^0, ..., p_{n+1}^0, y) \lambda(p_1^1, ..., p_{i-1}^1; s_i; p_{i+1}^0, ..., p_{n+1}^0, y) \mathrm{d}s_i \quad (6)$$

Two notes:

i. In general, a monotone transformation of the utility function changes the value of the left hand side. Thus, it must change the expression in (6).

The demand curves remain fixed but λ changes.

ii. It is always true that the integrals in (6) satisfy path independence. As noted earlier, any line integral in which the integrands have been derived by differentiating a real-valued function (like $v(p, p_m, y)$ above) will satisfy path independence. Otherwise it may not.

(e) Do not confuse (6) with the same expression after deleting λ:

$$- \sum_{i=1}^{n+1} \int_{p_i^0}^{p_i^1} x_i(p_1^1, ..., p_{i-1}^1; s_i; p_{i+1}^0, ..., p_{n+1}^0, y) \mathrm{d}s_i \quad (7)$$

i. It is always true that a monotone transformation of the utility function has no effect on (7). The expression involves only demand curves. What do the areas defined by (7) mean, though? Not much usually.

ii. In general, the line integral that gives rise to (7) is path dependent. Evaluate the integrals in a different order (or do not move parallel to the axes at all) and you will generally get a different result.
"The problem" with (7) isn't that it is path dependent, though. The problem is that after deleting λ it doesn't bear any connection to the change in utility or to anything else.

(f) The interest in (7) comes from the case in which preferences are homothetic. Then (7) does have a meaning, of sorts.
Using (5), take $\lambda(y)$ out of the integrals in (6) and the sum. This gives:

$$\frac{v(p^1, p_m^1, y) - v(p^0, p_m^0, y)}{\lambda(y)}$$

$$= -\sum_{i=1}^{n+1} \int_{p_i^0}^{p_i^1} x_i(p_1^1, ..., p_{i-1}^1; s_i; p_{i+1}^0, ..., p_{n+1}^0, y) \mathrm{d}s_i$$

i. The right hand side is (7).
It is still unaffected by monotone transformations of the utility function. Therefore so is the left hand side. The left hand side can therefore provide a meaningful interpretation to the right hand side: the monetary value of the total utility obtained from consuming a particular good.
This is Marshall's consumer's surplus – the monetary value of the total utility obtained from consuming a particular good.
We repeat that while this concept is now meaningful, it is superfluous.

ii. The left hand side makes no reference to any paths of integration (there are no integrals on the left hand side). Therefore, the right hand side is path independent.

(g) Example
CES utility, Cobb-Douglas utility.

8. Leontief preferences (vertical compensated demand)

   (a) Indifference curves

   $$U(x, m) = \min(a_1 x_1, ..., a_n x_n, a_m m)$$

   In the 2-good case, indifference curves are "*L*-shaped." All kink points occur on the ray from the origin with slope a_1/a_m.

   Figure 4

      i. To see this, fix an indifference curve:
         $\{(x_1, m) \mid \min(a_1 x_1, a_m m) = c\}$
         Partition the non-negative real plane into three sets:
         $\{(x_1, m) \mid a_1 x_1 = a_m m\}$
         $\{(x_1, m) \mid a_1 x_1 > a_m m\}$
         $\{(x_1, m) \mid a_1 x_1 < a_m m\}$
         The first set defines a ray from the origin with slope a_1/a_m.
         The second set defines all points to the *right* of that ray (x_1 is large at given m).
         The third set defines all points to the *left* of that ray.
      ii. On the second set, the indifference curve is all (x_1, m) such that $a_m m = c$. Therefore it is all x_1 such that $m = c/a_m$. This is a horizontal line at $m = c/a_m$ to the right of the ray.
      iii. On the third set, the indifference curve is all (x_1, m) such that $a_1 x_1 = c$. Therefore it is all m such that $x_1 = c/a_1$. This is a vertical line at $x_1 = c/a_1$.

   (b) Compensated demand

   (Inverse) compensated demand curves are vertical.

   Regardless of how prices change, the post-compensation budget constraint is tangent to the reference indifference curve at the kink point. Quantities in the compensated equilibrium therefore remain fixed as prices vary.

   (c) Slutsky equation

   The effect of a price change on standard demand comes entirely from the income effect. There is no substitution effect.

   (d) Excess burden

   There is no excess burden using the `LD` measure. That is the standard result.

   There is, however, using the *o*`R` measure!

9. Vertical regular demand

   (a) People sometimes assume that if a tax produces no change in *observed* behavior then it produces no excess burden.

       This is *wrong*.

       No change in observed behavior means that the *standard* demand curve is vertical. The income effect and substitution effect offset each other. However, the compensated demand curve could still be flat and the excess burden triangle large.

   (b) Slutsky equation

       All we can say is that, in an exogenous income model, the good must be inferior.

       The substitution effect is negative and the income effect is negative with equal magnitude. The total effect, from the Slutsky equation, is zero.

   (c) In an endogenous income (or endowment) model the good need not be inferior because you may be selling it, not buying it.

       This follows from the Slutsky equation for that model (see Silberberg).