Lecture 8

Outline

1. Characterizing total excess burden at given t: fixed utility measure
2. Single price change: trapezoids and triangles
3. Many prices change: trapezoids
4. Many prices change: not just triangles
5. Marginal excess burden with the fixed utility measure

1. Characterizing total excess burden at given t: fixed utility measure

   (a) Recall:

   $$\text{TEB}^{\texttt{LD}}(t) \;=\; I^o - E[q^o, V(q^o + t, I^o)] - tx(q^o + t, I^o)$$

   We now characterize total excess burden for some t through the "fixed utility" measure.

   The following characterizations are used over and over again in the literature.

   (b) Fix two (not necessarily distinct) tax vectors, say:

   $$t, \;\; \hat{t}$$

   We use the first one to define a reference utility level:

   $$V_t \equiv V(q^o + t, I^o)$$

   We use the second to define the state $\texttt{D}$:

   $$q^{\texttt{D}} = q^o + \hat{t}$$
   $$I^{\texttt{D}} = I^o$$

   We use these to define the following "characterization function":

   $$\text{CF}_t^{\texttt{LD}}(\hat{t}) = E(q^o + \hat{t}, V_t) - E(q^o, V_t) - \hat{t}x^c(q^o + \hat{t}, V_t) \tag{1}$$

   Note the following:

   i. If $\hat{t} = t$ then $\text{TEB}^{\texttt{LD}}(t) = \text{CF}_t^{\texttt{LD}}(t)$.

   ii. Note that $\text{CF}_t^{\texttt{LD}}(\hat{t})$ is defined using compensated demand. This is justified since our goal is simply to obtain *interpretations* of the total excess burden when $\hat{t} = t$.

(c) Recall Lecture 3. The most useful path for analytical purposes moves from one vector to the other by traveling parallel to each axis one dimension at a time. Let us call this path σ. We will not write it out in closed form. Instead, we will write it as the collection of sub-paths:

$$i = 0, ..., n$$

$$\sigma_i : [0, t_i] \to \Re^{n+1}$$

$$\sigma_i(\tau) = \begin{bmatrix} q_0^o + \hat{t}_0 \\ \vdots \\ q_{i-1}^o + \hat{t}_{i-1} \\ q_i^o + \tau \\ q_{i+1}^o \\ \vdots \\ q_n^o \end{bmatrix}$$

The following now holds (recall Lecture 3):

$$\begin{aligned}
&\mathrm{CF}_t^{\mathtt{LD}}(\hat{t}) \\
&= E(q^o + \hat{t}, V_t) - E(q^o, V_t) - \hat{t}x^c(q^o + \hat{t}, V_t) \\
&= \sum_{i=0}^{n} \int_0^{\hat{t}_i} \nabla E[\sigma_i(\tau), V_t]\sigma_i'(\tau)\mathrm{d}\tau - \hat{t}x^c(q^o + \hat{t}, V_t) \\
&= \sum_{i=0}^{n} \int_0^{\hat{t}_i} \nabla E[\sigma_i(\tau), V_t](0, ..., 0, 1, 0, ..., 0)\mathrm{d}\tau - \hat{t}x^c(q^o + \hat{t}, V_t) \\
&= \sum_{i=0}^{n} \int_0^{\hat{t}_i} x_i^c[\sigma_i(\tau), V_t]\mathrm{d}\tau - \hat{t}x^c(q^o + \hat{t}, V_t) \\
&= \sum_{i=0}^{n} \int_0^{\hat{t}_i} x_i^c(q_0^o + \hat{t}_0, ..., q_i^o + \tau, q_{i+1}^o, ..., q_n^o, V_t)\mathrm{d}\tau - \hat{t}x^c(q^o + \hat{t}, V_t) \\
&= \sum_{i=0}^{n} \int_0^{\hat{t}_i} x_i^c(q_0^{\mathtt{D}}, ..., q_i^o + \tau, q_{i+1}^o, ..., q_n^o, V_t)\mathrm{d}\tau - \hat{t}x^c(q^o + \hat{t}, V_t) \qquad (2)
\end{aligned}$$

Of course, this holds in particular when $\hat{t} = t$.

2. Single price change: trapezoids and triangles

   (a) Suppose only good i is taxed.

   Equation (2) reduces to a single integral.

   Note! The demands for other goods may change, but those markets are irrelevant to the computation if the prices for those goods do not change. This holds by our assumption of linear technology.

   Conversely, if prices in other markets change then it is not possible to compute the excess burden from a tax on good i by looking at the market for good i alone.

   (b) Since only good i is taxed, we can simplify the notation a bit by writing the compensated demand *curve* for good i by:

   $$x_i^c(q_i; q_{-i}^o, V_t)$$

   We then have:

   $$\mathrm{CF}_t^{\mathtt{LD}}(\hat{t}) = \int_0^{\hat{t}_i} x_i^c(q_i^o + \tau; q_{-i}^o, V_t)\mathrm{d}\tau - \hat{t}_i x_i^c(q_i^o + \hat{t}_i; q_{-i}^o, V_t)$$

   We further assume that x_i^c is linear in its own price (at least locally). Then:

   i. The integral itself defines the area of a trapezoid.
   ii. The integral less the tax revenue defines the area of a triangle.
   iii. The triangle itself has height $\hat{t}_i$ and base $-\hat{t}_i S_{ii}$, so the area is:

   $$-\frac{1}{2}(\hat{t})_i^2 S_{ii}$$

   S_{ii} depends on q_{-i}^o and V_t but is independent of q_i by linearity.

   This is easiest to see in a graph.

Figure 1

   (c) The integral itself defines the area of a trapezoid.

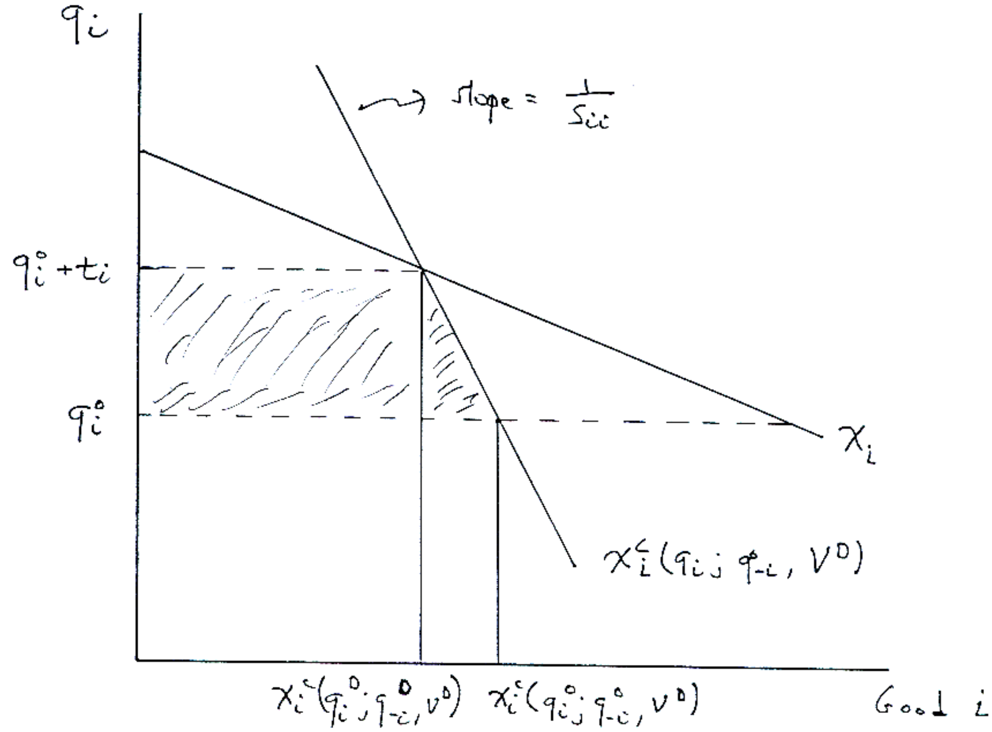
   By linearity:

   $$x_i^c(q_i; q_{-i}^o, V_t) = S_{ii} q_i + B_i$$

   where S_{ii} is the Slutsky term and B_i is the intercept.

   The integral i is then (we use $q_i^{\mathtt{D}}$ instead of $q_i^o + \hat{t}_i$ to ease the notation):

   $$\int_{q_i^o}^{q_i^{\mathtt{D}}} x_i^c(s_i; q_{-i}^o, V_t)\mathrm{d}s_i$$



Note: $\frac{\Delta q_i}{\Delta x_i^c} = \frac{1}{S_{ii}}$

$\Rightarrow \Delta x_i^c = S_{ii} \Delta q_i$

$\Rightarrow \Delta x_i^c = t_i \cdot S_{ii}$

(Δx_i^c ↓ base; t_i ↓ height)

Figure 1

$$
\begin{aligned}
&= \int_{q_i^o}^{q_i^{\mathtt{D}}} [S_{ii}s_i + B_i]\mathrm{d}s_i \\
&= \left[\frac{1}{2}S_{ii}s_i^2 + B_i s_i\right]_{q_i^o}^{q_i^{\mathtt{D}}} \\
&= \frac{1}{2}S_{ii}[(q_i^{\mathtt{D}})^2 - (q_i^o)^2] + B_i(q_i^{\mathtt{D}} - q_i^o) \\
&= (q_i^{\mathtt{D}} - q_i^o)\left[\frac{1}{2}S_{ii}(q_i^{\mathtt{D}} + q_i^o) + B_i\right] \\
&= (q_i^{\mathtt{D}} - q_i^o)\left[\frac{1}{2}S_{ii}q_i^{\mathtt{D}} + \frac{1}{2}B_i + \frac{1}{2}S_{ii}q_i^o + \frac{1}{2}B_i\right] \\
&= (q_i^{\mathtt{D}} - q_i^o)\left[\frac{x_i^c(q_i^{\mathtt{D}}; q_{-i}^o, V_t)}{2} + \frac{x_i^c(q_i^o; q_{-i}^o, V_t)}{2}\right]
\end{aligned}
$$

This is the formula for the area of a trapezoid.

(d) The integral less the tax revenue defines the area of a triangle.

Note that $\hat{t}_i = q_i^{\mathtt{D}} - q_i^o$. Then with the previous result:

$$
\int_{q_i^o}^{q_i^{\mathtt{D}}} x_i^c(s_i; q_{-i}^o, V_t)\mathrm{d}s_i - \hat{t}_i x_i^c(q_i^{\mathtt{D}}; q_{-i}^o, V_t)
$$

$$
\begin{aligned}
&= \frac{1}{2}\hat{t}_i\left[x_i^c(q_i^{\mathtt{D}}; q_{-i}^o, V_t) + x_i^c(q_i^o; q_{-i}^o, V_t)\right] - \hat{t}_i x_i^c(q_i^{\mathtt{D}}; q_{-i}^o, V_t) \\
&= \frac{1}{2}\hat{t}_i\left[x_i^c(q_i^o; q_{-i}^o, V_t) - x_i^c(q_i^{\mathtt{D}}; q_{-i}^o, V_t)\right] \qquad (3) \\
&= \frac{1}{2}\hat{t}_i[S_{ii}q_i^o + B_i - (S_{ii}q_i^{\mathtt{D}} - B_i)] \\
&= \frac{1}{2}\hat{t}_i[-\hat{t}_i S_{ii}] \qquad (4) \\
&= -\frac{1}{2}(\hat{t}_i)^2 S_{ii}
\end{aligned}
$$

Equation (3) gives the height and base of the triangle explicitly. Equation (4) says that the height of the triangle equals $\hat{t}_i$ and the base equals $-\hat{t}_i S_{ii}$.

(e) For any given tax rate, the triangle will tend to be larger the flatter the slope of the compensated demand curve. In the two-good case this curve is flat when the indifference curves are relatively flat.

This motivates the notion that the excess burden is, roughly speaking, proportional to the substitution effect. Large substitution effects are associated with large excess burdens.

(f) Excess burden increases with the *square* of the tax rate.

3. Many prices change: trapezoids

   If more than one price changes then this formula can be applied to each of the integrals. Equation (2) therefore says that we can obtain excess burden by summing up trapezoids market-by-market, giving total burden for the given t, and then subtracting tax revenue.

4. Many prices change: not just triangles

   (a) If more than one price changes then the link between equation (2) and "triangles" is more complicated. In particular, it is *not* valid to simply sum triangles market-by-market to obtain total excess burden The approach that worked in deriving (4) does not generalize (you could try it and see).

   (b) To obtain triangles exactly for finite changes in the tax rates, we know that at the very least we need compensated demand to be linear in own prices. Formally, we need S_{ii} constant in q_i for all i other than the numeraire.

   The following derivation makes the further assumption that all S_{ij} for $i \geq 0$ and $j \geq 0$ are constant in all prices $q_k, k \geq 0$. That is to say, all goods have constant cross price effects. The second-order Taylor's approximation to the function is then simple.[1]

   (c) The second-order Taylor's expansion gives:

$$\mathrm{CF}_t^{\mathtt{LD}}(\hat{t}) = \mathrm{CF}_t^{\mathtt{LD}}(0) + \left[\frac{\partial \mathrm{CF}_t^{\mathtt{LD}}}{\partial \hat{t}}(0)\right]' \hat{t} + \frac{1}{2}\hat{t}' \left[\frac{\partial^2 \mathrm{CF}_t^{\mathtt{LD}}}{\partial \hat{t} \partial \hat{t}}(0)\right] \hat{t}$$

   To evaluate this we need the gradient vector and the Hessian matrix for $\mathrm{CF}_t^{\mathtt{LD}}(\hat{t})$.

   From (1), the ith component of the gradient is:

$$\begin{aligned} \frac{\partial \mathrm{CF}_t^{\mathtt{LD}}(\hat{t})}{\partial \hat{t}_i} &= x_i^c(.) - x_i^c(.) - \hat{t}'(S_{0i}, S_{1i}, ..., S_{ni}) \\ &= -\sum_{k=0}^{n} \hat{t}_k S_{ki}, \quad i \geq 0 \end{aligned} \tag{5}$$

   This is evaluated at $\hat{t} = 0$, so all terms are zero.

[1] For the math, see Simon and Blume, page 832-33. Note that in this application, $a = 0$ and $h = \hat{t}$.

The ijth component of the Hessian is:

$$\begin{aligned}\frac{\partial^2 \mathrm{CF}_t^{\mathtt{LD}}(\hat{t})}{\partial \hat{t}_i \partial \hat{t}_j} &= -\frac{\partial}{\partial \hat{t}_j}\left(\sum_{k=0}^{n} \hat{t}_k S_{ki}\right) \\ &= -S_{ij},\ \ i \geq 0,\ j \geq 0\end{aligned}$$

This uses the assumption that all S_{ij} are constant in all prices, otherwise there would be more terms in these expressions. The Hessian of $\mathrm{CF}_t^{\mathtt{LD}}(\hat{t})$ reduces to the Slutsky matrix and the terms are all constants. There are no higher-order terms.

Therefore:

$$\begin{aligned}\mathrm{CF}_t^{\mathtt{LD}}(\hat{t}) &= 0 + (0, ..., 0)'\hat{t} + \frac{1}{2}\hat{t}'\,(-S)\,\hat{t} \\ &= -\frac{1}{2}\hat{t}'S\hat{t} \\ &= -\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{n}\hat{t}_i\hat{t}_j S_{ij}\end{aligned}$$

So, we have a quadratic form in the tax rates.

i. Terms of the form $(1/2)\hat{t}_i^2 S_{ii}$ are triangles.

ii. When more than one good is taxed there are cross terms. For example, with $n = 2$ we have:

$$\mathrm{CF}_t^{\mathtt{LD}}(\hat{t}) = -\left(\frac{1}{2}\hat{t}_0^2 S_{00} + \frac{1}{2}\hat{t}_1^2 S_{11} + \hat{t}_0\hat{t}_1 S_{01}\right)$$

iii. What effect is captured by the cross term? Suppose goods 1 and 2 are (Hicksian) substitutes, so the cross term is positive. Increasing $\hat{t}_2$ increases demand for good 1. This increases the revenue raised from good 1. This in itself tends to reduce the excess burden of the tax system.

iv. While the cross terms may in themselves tend to reduce the excess burden, the excess burden will not actually fall if $\hat{t}$ is optimal. If $\hat{t}$ is not optimal then it may fall.

5. Marginal excess burden with the fixed utility measure

Equation (5) provides a few classical results about commodity taxes.

Note! Throughout the tax literature, "marginal excess burden" is defined in ad hoc ways. It is sometimes constructed as a ratio of terms without any regard for whether there is an underlying "total" function of which it is the derivative. These expressions are difficult to interpret.

(a) First, the standard picture of marginal excess burden (one tax only).

Figure 2

(b) Marginal excess burden is zero if $\hat{t} = 0$.

Placing a "small" distortion in the kth market creates no excess burden. A common interpretation is, "the first marginal distortion is free."

(c) Marginal excess burden of a new tax may be very large *if there are other taxes present.*

Formally, the derivative in (5) evaluated at a tax vector that has $\hat{t}_i = 0$ may be large. The *levels* of all *other* taxes determine the extra excess burden from the new tax.

(d) Marginal excess burden from $\hat{t}_i$ increases with $\hat{t}_i$.

That is to say, total excess burden increases nonlinearly with the tax rate.

We saw this already with triangles, under the assumption the compensated demand curves were linear.

(e) The optimal tax vector equalizes the ratio of marginal excess burden from the tax to "marginal compensating revenue" from the tax.

Let t denote the optimal tax vector. Use this to define utility level V_t.

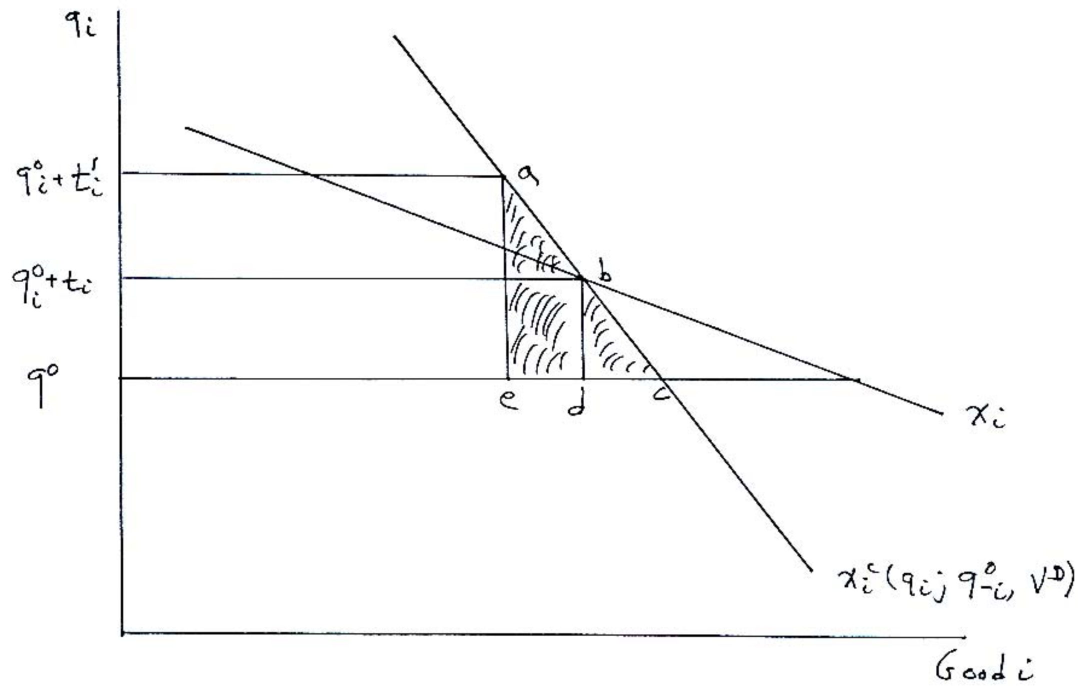
In deriving the Ramsey rule, we established:

$$-\frac{\sum_{k=0}^{n} t_k S_{ki}}{x_i^c(q^o + t, V_t)} = \theta, \quad i = 1, ..., n \tag{6}$$

From (5), the numerator is just the marginal excess burden using the fixed utility measure with $\hat{t} = t$. From the envelope theorem, the denominator is the extra revenue needed to hold utility constant (at V_t) as the tax on commodity i increases:

$$\frac{\partial \mathrm{CF}_t^{\mathrm{LD}}(\hat{t})/\partial \hat{t}_i}{\partial E(q^o + \hat{t}, V_t)/\partial \hat{t}_i} = \theta, \quad i = 1, ..., n$$

with $\hat{t} = t$. Let's call the denominator the "marginal compensating revenue." Then the Ramsey rule says that optimal taxes equate ratios of



$$CF_t^{LD}(t') - CF_t^{LD}(t) = ace - bcd$$

$$= abde$$

Figure 2

marginal excess burden to marginal compensating revenue across commodities.

(f) Marginal excess burden at the optimal tax vector is positive.

This follows from (6) if θ is positive, which we have already shown.

If t is not optimal, however, then total excess burden may fall from increasing a tax. This holds for the "same reason" that total excess burden is not just the sum of triangles.