Lecture 4

Outline

1. 3-Good Case (Corlett-Hague rule)
2. Optimal taxation with any CRS technology: the Ramsey Rule again
3. Production efficiency with public and private production (any CRS technology; following Auerbach, p. 100)

1. 3-Good Case (Corlett-Hague rule)

   The 3-good case (numeraire plus two taxed goods) provides a little insight into the structure of optimal taxes.

   (a) With three goods, the Ramsey rule gives us:

   $$\sum_{i=1}^{2} t_i S_{ik} = -\theta x_k, \quad k = 1, 2$$

   Writing these conditions in matrix form:

   $$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = -\theta \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

   Using Cramer's Rule:

   $$t_1 = -\theta \frac{1}{\Delta} \begin{vmatrix} x_1 & S_{12} \\ x_2 & S_{22} \end{vmatrix}$$

   $$t_2 = -\theta \frac{1}{\Delta} \begin{vmatrix} S_{11} & x_1 \\ S_{21} & x_2 \end{vmatrix}$$

   where

   $$\Delta = \begin{vmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{vmatrix}$$

   (b) We know from the analysis of θ after the derivation of the Ramsey rule that $\theta \geq 0$. We will suppose

   $$\theta > 0$$

   (c) The (full) Slutsky matrix is:

   $$S = \begin{bmatrix} S_{00} & S_{01} & S_{02} \\ S_{10} & S_{11} & S_{12} \\ S_{20} & S_{21} & S_{22} \end{bmatrix}$$

Assume that S is negative semi-definite. This implies:

$$\Delta \geq 0$$

The result follows from the fact that Δ is a 2nd order *principal minor* of S. The k^{th} order principal minor of a symmetric negative semi-definite matrix is nonnegative if k is even and less than or equal to zero ("nonpositive"?) if k is odd. See Simon and Blume (1994), Theorem 16.2 (p. 383).

i. Recall that if A is an $n \times n$ matrix, then the k^{th} order *principal minor* is the determinant of the $k \times k$ submatrix formed by deleting any $n-k$ columns and the corresponding rows.

ii. Myles says $\Delta \leq 0$ (his p. 124) while Auerbach says $\Delta > 0$ (his p. 92). Auerbach is correct, under the assumption the matrix is negative definite and not just semi-definite.

iii. You can replace "semi-definite" with "definite" and the weak inequalities with strict inequalities in the above theorem.
With definite matrices we usually focus on the *leading* principal minors. Any principal minor can be made into a leading principal minor of some matrix by permuting corresponding rows and columns. These permutations will not affect the definiteness of the matrix.
Here's a quick proof, just for the love of it. If E is the identity matrix with certain rows permuted then E' is the identity matrix with the corresponding columns permuted, EA is the matrix A with the same rows permuted, and AE' is the matrix A with the corresponding columns permuted. If $x'Ax > 0$ for all $x \neq 0$, then given any $y \neq 0$ we must have $y'(EAE')y = (y'E)A(E'y) = (E'y)'A(E'y) > 0$ so EAE' is also positive definite.

(d) Returning to the formulas and using the fact:

$$-\frac{\theta}{\Delta} < 0$$

gives:

$$\begin{aligned} t_1 > t_2 \quad &\longleftrightarrow \quad \begin{vmatrix} x_1 & S_{12} \\ x_2 & S_{22} \end{vmatrix} < \begin{vmatrix} S_{11} & x_1 \\ S_{21} & x_2 \end{vmatrix} \\ &\longleftrightarrow \quad S_{22}x_1 - S_{12}x_2 < S_{11}x_2 - S_{21}x_1 \end{aligned}$$

Without loss we can measure quantities at the solution so that all post-tax prices are 1. Then the homogeneity of compensated demand gives:

$$S_{10} + S_{11} + S_{12} = 0$$

$$S_{20} + S_{21} + S_{22} = 0$$

Use these to eliminate S_{12} and S_{21} respectively, then divide both sides by $x_1 x_2$:

$$\begin{aligned} t_1 > t_2 \quad &\longleftrightarrow \quad S_{22}x_1 + (S_{10}+S_{11})x_2 < S_{11}x_2 + (S_{20}+S_{22})x_1 \\ &\longleftrightarrow \quad \frac{S_{22}}{x_2} + \frac{S_{10}}{x_1} + \frac{S_{11}}{x_1} < \frac{S_{11}}{x_1} + \frac{S_{20}}{x_2} + \frac{S_{22}}{x_2} \\ &\longleftrightarrow \quad \frac{S_{10}}{x_1} < \frac{S_{20}}{x_2} \\ &\longleftrightarrow \quad \epsilon_{10}^c < \epsilon_{20}^c \end{aligned}$$

(e) Conclusion: at the optimum, the higher tax rate falls on the good with the smaller compensated cross-elasticity with the untaxed good.

(f) This is sometimes expressed as, "the higher tax rate falls on the good that is the relatively stronger compensated (or 'Hicksian') complement with the untaxed good."

This must be interpreted carefully. If for example:

$$0 < \epsilon_{10}^c < \epsilon_{20}^c$$

then both goods are substitutes with the numeraire, it's just that good 1 is less strong a substitute.

(g) It is not correct to interpret the rule as saying that taxing the relative complement of the numeraire is an attempt to overcome "the restriction" that we can't tax the numeraire. The restriction $t_0 = 0$ is without any loss of generality.

A more interesting question is whether taxing the relative complement of the numeraire is an attempt to overcome the restriction that we can tax only net trades and not endowments. Note that allowing all components of t to be chosen, or allowing $t_0 > 0$ and requiring some other $t_0 = 0$, would still not indirectly tax any endowments. I will leave this as an analytical puzzle.

Myles argues that this result is really just an artifact of the homogeneity condition. It is not entirely clear what he means by this.

2. Optimal taxation with any CRS technology: the Ramsey Rule again

(a) Recall the general optimal tax problem:

$$\begin{aligned} &\underset{q_1, \ldots, q_n}{\text{Max}} \; V(q) \\ &\text{subject to:} \quad F[x(q) + x^G] = 0 \end{aligned}$$

(b) Before solving this, we need a result from profit maximization. Recall that aggregate output solves:

$$\max_{y} \; py \quad \text{s.t.} \quad F(y) = 0$$

Using γ for the Lagrange multiplier:

$$\mathcal{L} = py + \gamma[F(y)]$$

Therefore:

$$\frac{\partial \mathcal{L}}{\partial y_i} = p_i + \gamma \frac{\partial F}{\partial y_i} = 0, \quad i = 0, ..., n$$

so:

$$p_i = -\gamma \frac{\partial F}{\partial y_i}, \quad i = 0, ..., n$$

If we take the ratio with the first equation then we have the n conditions:

$$\frac{p_i}{p_0} = \frac{\partial F / \partial y_i}{\partial F / \partial y_0}, i = 1, ..., n$$

Using both $\frac{\partial F}{\partial y_0} = 1$ (recall Lecture 2) and $p_0 = 1$ gives:

$$p_i = \frac{\partial F}{\partial y_i}, \quad i = 1, ..., n$$

(actually, it holds at $i = 0$ as well).

Attachment

$$\mathcal{L} = V(q) + \mu F[X(q) + x^G]$$

$$\frac{\partial \mathcal{L}}{\partial q_k} = \frac{\partial V}{\partial q_k} + \mu \left[F_0 \cdot \frac{\partial X_0}{\partial q_k} + \cdots + F_n \frac{\partial x_n}{\partial q_k} \right]$$

$$= 0 \quad , \quad k = 1, \ldots, n \quad ; \quad F_i = \partial F / \partial y_i$$

From the individual's utility maximization problem: $\frac{\partial V}{\partial q_k} = -\alpha X_k$, so

$$\Rightarrow \alpha X_k = \mu \sum_{i=0}^{n} F_i \frac{\partial X_i}{\partial q_k} \quad , \quad k = 1, \ldots, n$$

$$= \mu \sum_{i=0}^{n} p_i \cdot \frac{\partial X_i}{\partial q_k}$$

$$= \mu \sum_{i=0}^{n} (q_i - t_i) \frac{\partial X_i}{\partial q_k}$$

$$= \mu \left[\sum_{i=0}^{n} q_i \frac{\partial X_i}{\partial q_k} - \sum_{i=1}^{n} t_i \frac{\partial X_i}{\partial q_k} \right]$$

By CRS, $q \cdot X = 0$, so differentiating this with q_k gives:

$$X_k + \sum_{i=0}^{n} q_i \cdot \frac{\partial X_i}{\partial q_k} = 0 \qquad k = 1, \ldots, n$$

Substituting:

$$\alpha X_k = \mu \left[- X_k - \sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial q_k} \right]$$

Rearrange:

$$\sum_{i=1}^{n} t_i \frac{\partial X_i}{\partial q_k} = - \left(\frac{\mu + \alpha}{\mu} \right) X_k$$

$, k = 1, \ldots, n$

This has exactly the same form as the equation we derived with linear technology, just before we used the Slutsky equation.

$\Rightarrow$ we again obtain the equal percentage change rule.

(c) We therefore have once again:

$$\frac{\sum_{i=1}^{n} t_i S_{ik}}{x_k} = \text{terms independent of } k, \quad k = 1, ..., n$$

(d) This has a similar equal percentage change interpretation as before, but it is not quite identical.

i. The integrals are still evaluated from q^0 to $q^0 + t$.

ii. The new equilibrium prices, q^1, are *not* $q^0 + t$ unless the tax is fully shifted forward as before.

iii. Nevertheless, the formula says that compensated demand between the original prices and $q^0 + t$ (not the new equilibrium prices) must change by the same percentage for all goods.

3. Production efficiency with public and private production (any CRS technology; following Auerbach, p. 100).

(a) Recall, production is efficient if, for every pair of factors, the ratio of their marginal products is the same in all lines of production.

It is then not possible to rearrange factors across lines of production to increase one output without decreasing another.

(b) The production efficiency lemma says that under CRS, the optimal commodity tax vector will maintain production efficiency.

"The basic intuition is that as long we can tax all but one of the commodities, we can bring about any possible configuration of relative prices consistent with a given level of revenue." (Auerbach).

Thus, given a vector of prices that raise the required revenue without production efficiency, it would be possible to raise the same revenue, have more of some or all goods, and thereby increase the consumer's utility.

(c) This implies the following:

i. Even if the government could levy partial factor taxes, it wouldn't. The government would not cause different firms to face different input prices.

ii. Taxing an intermediate good in a particular industry would be equivalent to a partial factor tax. So, even if the government could do this, it wouldn't.

(d) Now suppose that the government is also a producer. It purchases factors on the open market, uses its own technology to produce goods, and then sells the goods.

This is in addition to still buying and donating x^G to consumers. We keep the two problems unlinked – linking them seems to generate a somewhat more complicated problem.

Is production still efficient?

(e) Let s denote the government's netput vector and $G(s) = 0$ its transformation function.

Note that we assume that the government chooses s to maximize the welfare of the consumer. It does not act like a private firm, choosing s to maximize profits taking p as given.

Are these different problems? There must be a literature about this!

(f) Any gap between the cost of what the government buys and the revenue from what it sells increases the tax revenue that must be raised.

If revenue exceeds cost then $ps > 0$, so this is subtracted from the revenue requirement. Similarly, if cost exceeds revenue, then $ps < 0$, and this is also subtracted from the revenue requirement.

Market clearance is now:

$$x(q) + x^G = y(p) + s$$

The government's budget constraint is:

$$(q - p)[x(q) + x^G] = qx^G - ps$$

which reduces to:

$$(q - p)x(q) = px^G - ps$$

As before, Walras law for the model shows that the government's budget constraint is redundant.

i. It is always worth checking the accounting.
Premultiply market clearing by p and use $py(p) = 0$:

$$px(q) + px^G = py(p) + ps = ps$$

so:

$$-px(q) = px^G - ps$$

Using $qx(q) = 0$:

$$qx(q) - px(q) = (q - p)x(q) = px^G - ps$$

(g) The optimization problem is now:

$$\underset{q_1,\dots,q_n;\, s_0, s_1,\dots,s_n}{\text{Max}} \; V(q)$$

$$\text{subject to:} \quad F[x(q) + x^G - s] = 0, \quad G(s) = 0$$

(h) Production should be efficient. For any two factors i and j we should have:

$$\frac{\partial F/\partial y_i}{\partial F/\partial y_j} = \frac{\partial G/\partial y_i}{\partial G/\partial y_j}$$