Lecture 3

Outline

Optimal commodity tax problem with a single consumer

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1. The general problem

   The government's budget constraint is:

   $$(q-p)[x[q,\pi(p)]+x^G] = qx^G$$

   The left hand side is the tax revenue the government gathers. The right hand side is the cost of its purchases.

   (a) Note that the government cannot satisfy this constraint if people engage in no net trades. If people just consume their endowments, then $(q-p)(x^G) = qx^G$ so $-px^G = 0$. This is impossible since all terms in x^G are positive: the government makes only purchases, it has no endowment to sell.

   If people were inclined to consume their endowments, the optimal tax vector would have to induce them not to.

   (b) The general problem is easier to handle if we rewrite this to eliminate x^G from the left hand side. This gives $(q-p)x[q,\pi(p)] = px^G$.

   The most general way of writing the optimal commodity tax problem is then:

   $$\begin{aligned} &\max_{q_0,\ldots,q_n;\,p_0,\ldots,p_n} V[q,\pi(p)] \\ &\text{subject to:} \quad x_i(q)+x_i^G = y_i(p), \quad i=0,\ldots,n \\ &\qquad\qquad\quad (q-p)x[q,\pi(p)] = px^G \end{aligned}$$

   This, however, involves some redundancy and can be substantially simplified.

2. Walras Law

   Pre-multiply the set of equilibrium conditions by the producer price vector. This gives:

   $$px[q, \pi(p)] + px^G = py(p)$$

   We have:

   $$py(p) = \pi(p) = qx[q, \pi(p)]$$

   The first equality is from the definition, the second is from the individual budget constraint. So:

   $$px[q, \pi(p)] + px^G = qx[q, \pi(p)]$$

   Therefore:

   $$px^G = qx[q, \pi(p)] - px[q, \pi(p)] = (q - p)x[q, \pi(p)]$$

   Thus, the government's budget constraint is redundant if we have all $n + 1$ equilibrium conditions.

   In other words:

   (a) We can drop the government's budget constraint from the problem if we have all $n + 1$ equilibrium conditions.

   This is how it is usually used.

   (b) Alternatively, we can drop one of the equilibrium conditions if we include the government's budget constraint.

3. Tax vector normalization (uses CRS)

   Under the assumption of CRS (and the restriction to taxing only net trades), one of the tax rates is redundant.

   (a) To see the intuition, recall the labor-income model with profit income. Consumption good is the numeraire. Suppose the government places a tax on net trades of both labor supply and consumption. Let w be the producer price of labor This is the gross wage since the producer buys labor. We model what occurs if the gross wage falls by the full amount of the tax. Then:

   $$x = \begin{bmatrix} -L \\ Y \end{bmatrix} \qquad q = p + t = \begin{bmatrix} w + t_0 \\ 1 + t_1 \end{bmatrix}$$

The budget constraint is $qx = \pi(p)$, so in this case:

$$(w + t_0)(-L) + (1 + t_1)Y = \pi(p)$$

Rearranging gives:

$$Y = \frac{\pi(p)}{1 + t_1} - \left(\frac{w + t_0}{1 + t_1}\right)(-L)$$

Because of profit income, the government needs both tax instruments to fully control the budget constraint.

If the profit income were missing, however, the government could fully control the constraint with just one tax.

Under CRS (given that we are taxing just net trades), one of the taxes is redundant.

(b) More formally now.

Suppose there is profit income. Then indirect utility and demands depend on both prices and income:

$$V[q, \pi(p)], \quad x[q, \pi(p)]$$

These are homogeneous of degree zero in (q, p) jointly. consumer and producer prices *jointly*. If we double all producer prices then profits double (the profit function is homogenous of degree 1 in all prices), so if we also double consumer prices then indirect utility and demands are unchanged (they are homogeneous of degree zero in prices and income).

Supply still depends on producer prices alone:

$$y[(p)]$$

Supply is homogenous of degree zero in producer prices.

(c) We can arbitrarily set the consumer price of one good equal to "1" or the producer price of one good equal to "1," *but not both in general.*

Suppose $q_0 > 1$ and we rescale all consumer *and producer* prices by q_0. Then indirect utility, demands, and supply are unchanged.

We now have $q_0' = 1$.

(d) If $\pi(p) = 0$ we can also independently rescale the producer prices. Indirect utility and demands no longer depend on $\pi(p)$. Supply is unaffected by rescaling all prices.

Under CRS, then, and without any loss of generality we can assume $p_0' = 1$. Therefore:

$$q_0' = p_0' = 1$$

which means that without any loss of generality:

$$t_0 = 0$$

(e) *Note!* The untaxed commodity may be a good for which we have an endowment. There is still no loss of generality!

The fact that we do not tax a commodity with an endowment does not imply we are foregoing a lump-sum tax. This was already implied by our restriction to taxing just net trades.

What has just been shown is that, given the restriction to taxing net trades, there is no further loss of generality (under CRS) to not taxing the good with an endowment at all.

4. Problem I: general CRS technology

From our previous results, in the general problem:

(a) We know $q_0 = 1$ and $p_0 = 1$, so these do not appear as choice variables.

(b) We can eliminate the government's budget constraint.

If we then substitute the market clearing conditions into the technology the producer prices no longer appear. We therefore have:

$$\max_{q_1, \ldots, q_n} V(q)$$
$$\text{subject to: } F[x(q) + x^G] = 0$$

This problem determines the demands that must be met. Given this, the government chooses producer prices so that firms will supply the required amount. Those producer prices are determined by the demands and the technological relationships derived earlier:

$$p_i = p_0 \frac{\partial F / \partial y_i}{\partial F / \partial y_0} = \frac{\partial F}{\partial y_i}, i = 1, \ldots, n$$

where we now use $p_0 = 1$. We will return to this problem after considering the case of linear technology.

5. Problem II: linear technology

(a) With a linear technology each derivative $\frac{\partial F}{\partial y_i}$ is a constant. These constants then determine the equilibrium producer prices through:

$$p_i = \frac{\partial F}{\partial y_i}, i = 1, \ldots, n$$

So, technology determines the producer prices and these are fixed.

It must be the case that all factor demand curves and all produced goods supply curves are horizontal.

This is what Karl Marx was after – prices are determined independently of demand!

Denote the equilibrium producer price vector by:

$$p^*$$

(b) Regarding linear technology:

i. In general equilibrium, CRS alone does not imply a linear technology.

ii. If labor is the only factor in production and all technologies are CRS, then the production possibilities frontier (as a relationship among the produced goods) is linear in every direction.

iii. For more general results on linear technology, see Kemp et al. (1978).

(c) If all producer prices are fixed, then consumer prices are the producer prices plus taxes:

$$q = p^* + t$$

With p^* given and q determined by t, the market clearing conditions can be dropped from the original problem. All that remains is the government's budget constraint.

Since both p^* and x^G are exogenous, we can eliminate p^*x^G in the constraint and replace it with R, "revenue."

i. *Note!* If producer prices change then we can not do this. A given quantity of numeraire would purchase different quantities of goods depending on equilibrium prices.

ii. Of course, if the government were not providing a vector of goods, but were merely making a lump-sum return of numeraire, then we would have a revenue constraint with just CRS.

The problem in which the government levies commodity taxes on a single individual to make a lump-sum return of numeraire is less interesting than the problem we consider.

(d) Our problem now becomes:

$$\max_{t_1,\ldots,t_n} V(p^* + t)$$
$$\text{subject to:} \qquad tx(p^* + t) = R$$

6. Ramsey rule for the linear technology model

(a) The derivation is attached.

Attachment.

Two points to note:

i. We have $V(p^* + t) \equiv V(p_0^* + t_0, ..., p_n^* + t_n)$ (of course $t_0 = 0$), so using the fact that t_k appears only in the kth entry and enters additively:

$$\frac{\partial V(p^* + t)}{\partial t_k} = \frac{\partial V}{\partial q_k}\frac{\partial q_k}{\partial t_k} = \frac{\partial V}{\partial q_k}$$

ii. Keep in mind that we have shown that at the optimal tax vector:

$$\frac{\sum_{i=1}^{n} S_{ik} t_i}{x_k} < 0, \quad k = 1, ..., n$$

This comes up again and again.

7. Interpretation of the Ramsey rule

(a) We need to analyze $\sum_{i=1}^{n} S_{ik} t_i$, where:

$$S_{ik} = \frac{\partial x_i^c}{\partial q_k} = \frac{\partial x_k^c}{\partial q_i} = S_{ki}$$

(b) The following mathematics comes up again and again. The review is worthwhile.

At the most general level, we are given two vectors of consumer prices $q(0)$, and $q(t)$. Intuitively, a path between the two is a mapping:

$$\sigma : [0, b] \rightarrow \Re^{n+1}$$

with:

$$\sigma(0) = q(0), \sigma(b) = q(t)$$

$$\sigma(\tau) = (q_0(\tau), q_1(\tau), ..., q_n(\tau))$$

The most useful path for analytical purposes moves from one vector to the other by traveling parallel to each axis one dimension at a time. Let us call this path σ. We will not write it out in closed form. Instead, we will write it as the collection of sub-paths:

$$\max_{t_1,\dots,t_n} \quad V(p^*+t)$$

$$\text{s.t.} \quad \sum_{i=1}^{n} t_i X_i = R$$

$$\mathcal{L} = V(p^*+t) + \lambda\left(\sum_{i=1}^{n} t_i X_i - R\right)$$

$$\frac{\partial V}{\partial q_k} = -\lambda\left(X_k + \sum_{i=1}^{n} t_i \frac{\partial X_i}{\partial q_k}\right), \quad k=1,\dots,n$$

Roy's Theorem: $\dfrac{\partial V}{\partial q_k} = -\alpha X_k$

$$\Rightarrow \quad -\alpha X_k = -\lambda X_k - \lambda \sum_{i=1}^{n} t_i \frac{\partial X_i}{\partial q_k}$$

$$\Rightarrow \quad \sum_{i=1}^{n} t_i \frac{\partial X_i}{\partial q_k} = -\left(\frac{\lambda-\alpha}{\lambda}\right) X_k, \quad k=1,\dots,n$$

X_k is demand <u>post-tax</u>: $X_k(p^*+t)$

The compensated demand curve at this point is $X(p^*+t, \underbrace{V(p^*+t)}_{U^*})$.

The slopes are linked by the Slutsky equation.

$$\frac{\partial X_i}{\partial q_k} = S_{ik} - X_k \cdot \frac{\partial X_i}{\partial I} \qquad \text{(evaluated at } I = 0\text{)}$$

$$\Rightarrow \sum_{i=1}^{n} t_i \left(S_{ik} - X_k \frac{\partial X_i}{\partial I} \right) = -\left(\frac{\lambda - \alpha}{\lambda} \right) X_k \quad , k = 1, \ldots, n$$

$$\Rightarrow \sum_{i=1}^{n} t_i S_{ik} = -\underbrace{\left(1 - \frac{\alpha}{\lambda} - \sum_{i=1}^{n} t_i \frac{\partial X_i}{\partial I} \right)}_{\theta} X_k , \quad k = 1, \ldots, n$$

(Myles has a ~~typo~~: compare (4.15) and (4.16)!)

$$\Rightarrow \boxed{\frac{\sum_{i=1}^{n} t_i S_{ik}}{X_k} = -\theta , \qquad k = 1, \ldots, n}$$

The sign of θ:

We have $\sum_{i=1}^{n} t_i S_{ik} = -\theta X_k, \quad k = 1, \dots, n$

Multiply both sides by t_k and sum over k:

$$\Rightarrow \sum_{k=1}^{n} t_k \sum_{i=1}^{n} t_i S_{ik} = -\theta \sum_{k=1}^{n} t_k X_k$$

Since $t_0 = 0$ we have $t_0 \cdot \sum_{i=1}^{n} t_i S_{i0} = 0$.

Adding this to the left side gives:

$$\Rightarrow \sum_{k=0}^{n} t_k \sum_{i=1}^{n} t_i S_{ik} = -\theta \sum_{k=1}^{n} t_k X_k$$

Similarly, $\sum_{k=0}^{n} t_k \cdot t_0 \cdot S_{0k} = 0$. Adding this in gives:

$$\Rightarrow \underbrace{\sum_{k=0}^{n} t_k \sum_{i=0}^{n} t_i S_{ik}}_{} = -\theta \sum_{k=1}^{n} t_k X_k$$

$\Leftrightarrow t' S t$, quadratic form in S.
S is negative semi-definite,
since it is the Hessian of the expenditure function.

Therefore $-\theta \sum_{k=1}^{n} t_k X_k \leq 0$

We also know $\sum_{k=1}^{n} t_k X_k = R > 0$,

Since t is the optimal tax vector and it was chosen to satisfy this constraint.

$\therefore \quad -\theta \leq 0$

or $\theta \geq 0$

$$i = 0, ..., n$$

$$\sigma_i : [0, t_i] \to \Re^{n+1}$$

$$\sigma_i(\tau) = \begin{bmatrix} q_0(t) + t_0 \\ \vdots \\ q_{i-1}(t) + t_{i-1} \\ q_i(t) + \tau \\ q_{i+1}(t) \\ \vdots \\ q_n(t) \end{bmatrix}$$

Assuming fixed producer prices, this is just:

$$i = 0, ..., n$$

$$\sigma_i : [0, t_i] \to \Re^{n+1}$$

$$\sigma_i(\tau) = \begin{bmatrix} p_0^* + t_0 \\ \vdots \\ p_{i-1}^* + t_{i-1} \\ p_i^* + \tau \\ p_{i+1}^* \\ \vdots \\ p_n^* \end{bmatrix}$$

Using basic theorems for line integrals of vector fields that happen to be gradients of some function, we have (note that we use $t_0 = 0$):

$$
\begin{aligned}
x_k^c(p^* + t) - x_k^c(p^*) &= \int_\sigma \nabla x_k^c \mathrm{d}s \\
&= \sum_{i=1}^n \int_{\sigma_i} \nabla x_k^c \mathrm{d}s \\
&= \sum_{i=1}^n \int_0^{t_i} \nabla x_k^c[\sigma_i(\tau)]\sigma_i'(\tau)\mathrm{d}\tau \\
&= \sum_{i=1}^n \int_0^{t_i} \nabla x_k^c[\sigma_i(\tau)](0, ..., 0, 1, 0, ..., 0)\mathrm{d}\tau \\
&= \sum_{i=1}^n \int_0^{t_i} \frac{\partial x_k^c(\sigma_i(\tau))}{\partial q_i}\mathrm{d}\tau \\
&= \sum_{i=1}^n \int_0^{t_i} \frac{\partial x_k^c(p_0^* + t_0, ..., p_i^* + \tau, p_{i+1}^*, ..., p_n^*)}{\partial q_i}\mathrm{d}\tau \\
&= \sum_{i=1}^n \int_0^{t_i} S_{ki}(p_0^* + t_0, ..., p_i^* + \tau, p_{i+1}^*, ..., p_n^*)\mathrm{d}\tau \\
&= \sum_{i=1}^n S_{ki} \int_0^{t_i} \mathrm{d}\tau \\
&= \sum_{i=1}^n t_i S_{ki} \\
&= \sum_{i=1}^n t_i S_{ik}
\end{aligned}
$$

Notice:

i. The point at which the integrands are evaluated moves as we move along the path.

ii. The third-to-last step uses the assumption that the S_{ki} are constant in the relevant range.

iii. The last step uses the symmetry of the Slutsky matrix.

(c) As an interesting aside, how far would we get with regular demand?

Recall that the optimal tax problem gives us the expression:

$$\sum_{i=1}^n t_i \frac{\partial x_i}{\partial q_k}$$

The line integral for regular demand would give us:

$$x_k(p^* + t) - x_k(p^*) = \sum_{i=1}^n \int_{\sigma_i} \nabla x_k \mathrm{d}s$$

$$= \sum_{i=1}^{n} t_i \frac{\partial x_k}{\partial q_i}$$

The two expressions are not the same. Regular demand need not have the kind of symmetry that would allow us to convert the latter expression into the former. So, we cannot use the latter to interpret the former.

The potential lack of symmetry reflects the fact that the vector of regular demands, $(x_0, ..., x_n)$, is not *itself* the gradient of something. In particular it is not the gradient of indirect utility. This is $(-\alpha x_0, ..., -\alpha x_n)$, where α is the marginal utility of income.

(d) Using all of the earlier results:

$$\frac{x_k^c(p^* + t, U^*) - x_k^c(p^*, U^*)}{x_k^c(p^* + t, U^*)} = \frac{\sum_{i=1}^{n} S_{ki} t_i}{x_k^c(p^* + t, U^*)} = -\theta < 0, \quad k = 1, ..., n$$

It follows that, for any pair of goods k and l:

$$\frac{x_k^c(p^* + t, U^*) - x_k^c(p^*, U^*)}{x_k^c(p^* + t, U^*)} = \frac{x_l^c(p^* + t, U^*) - x_l^c(p^*, U^*)}{x_l^c(p^* + t, U^*)}$$

This gives the Ramsey rule:

> *If all S_{ij} are constant in the relevant range, then if t is an optimal commodity tax vector, then eliminating all taxes should cause an equal percentage change in the compensated demand for all goods.*

Notice that the post-tax situation is the reference point since this is used in the denominator. Technically the change is from the post-tax vector to the pre-tax vector, not the other way around.

(e) In general, optimal taxation implies intervening in every market and at a different tax rate.

This leads to a presumption against the efficiency of broad based taxes.

On the other hand, there are models of government behavior (positive political economy) that suggest that real governments should perhaps be restricted to broad based taxes since they choose rates to solve a different problem from the one stated above.

(f) The formula should not be interpreted as saying that, in the real world, necessities "should" be heavily taxed. The real world has many additional properties including the fact that there is more than one person. While the presumption is that there would be an efficiency loss from *not* taxing necessities, there may be an equity gain that produces a net gain in social welfare.

(g) Since compensated demand for good k depends on *all* prices, it is not true that goods that are more own-price inelastic necessarily get higher taxes.

However, if we assume that these cross effects are zero, then we do obtain this result.

8. Inverse elasticity rules

   (a) Version for Compensated Demand

      i. Recall from the theory of the consumer Hicks' three laws:

$$S_{ik} = S_{ki}$$
$$S_{kk} < 0$$
$$\sum_{i=0}^{n} S_{ik} q_i = 0$$

      The last follows immediately from the fact that compensated demand curves are homogeneous of degree zero in prices.

      ii. The last two imply that we cannot assume that *all* cross effects are zero. For example, if there are just two goods "0" and "1", they must be Hicksian substitutes ($S_{01} > 0$).
      However, there is no contradiction in assuming that the only cross effects that are nonzero are with the numeraire. Thus:

$$S_{ik} = 0, \quad i = 1, ..., n, \quad k = 1, ..., n, \quad i \neq k$$
$$\sum_{i=0}^{n} S_{ik} q_i = S_{kk} q_k + S_{0k} = 0, \quad k = 1, ..., n$$

      iii. This is a substantive behavioral assumption about cross effects with a particular good. It may be more empirically reasonable for some goods than others. The choice of numeraire "matters" if we are going to express a behavioral restriction in terms of it.

      iv. The derivation of the Ramsey rule gave us for good k:

$$\sum_{i=1}^{n} t_i S_{ik} = -\theta x_k^c, \quad k = 1, ..., n$$

      If the only cross effects are with the numeraire, then:

$$t_k S_{kk} = -\theta x_k^c, \quad k = 1, ..., n$$

      Rewriting S_{kk} as the derivative of compensated demand and multiplying both sides by q_k gives:

$$\frac{\partial x_k^c}{\partial q_k} \frac{q_k}{x_k^c} \equiv \epsilon_{kk}^c = -\theta \frac{q_k}{t_k}$$

      The elasticity is independent of all prices other than q_k, and it is evaluated at post-tax prices and post-tax utility U^*.
      Taking reciprocals and using $q_k = p_k^* + t_k$:

$$\frac{1}{\epsilon_{kk}^c} = -\frac{1}{\theta} \frac{t_k}{p_k^* + t_k}$$

Thus:

$$\frac{1}{-\epsilon^c_{kk}} = \frac{1}{\theta}\frac{t_k}{p^*_k + t_k} > 0$$

and the right hand side is increasing in t_k. It follows that the smaller the absolute value of the compensated demand elasticity (the "steeper" the compensated demand curve) the higher the optimal tax rate:

$$t_k \propto \frac{1}{-\epsilon^c_{kk}}$$

(b) Version for Regular Demand

i. We start with the (perfectly reasonable) assumption that own-price effects are non-zero:

$$\frac{\partial x_k}{\partial q_k} \neq 0$$

With CRS demands are homogenous of degree zero in prices, so:

$$\sum_{i=0}^{n} \frac{\partial x_k}{\partial q_i} q_i = 0$$

Again, we cannot also assume that *all* cross effects are zero.

ii. As before, we can as a logical (as opposed to empirical) matter assume that the only cross effects that are non-zero are with the numeraire. Thus:

$$\frac{\partial x_k}{\partial q_i} = 0, \quad i = 1, ..., n, \quad k = 1, ..., n, \quad i \neq k$$

$$\sum_{i=0}^{n} \frac{\partial x_k}{\partial q_i} q_i = \frac{\partial x_k}{\partial q_k} q_k + \frac{\partial x_k}{\partial q_0} = 0$$

iii. The derivation of the Ramsey rule gave us for good k:

$$\sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial q_k} = -\left(\frac{\lambda - \alpha}{\lambda}\right) x_k, \quad k = 1, ..., n$$

With the restriction above this reduces to:

$$t_k \frac{\partial x_k}{\partial q_k} = -\left(\frac{\lambda - \alpha}{\lambda}\right) x_k, \quad k = 1, ..., n$$

The same manipulations as above will then give:

$$t_k \propto \frac{1}{\epsilon_{kk}}$$