Lecture 2

Outline

1. Leisure-income and labor-income models
2. Basic structure of the optimal tax model
3. Government
4. Consumers
5. Production
6. A digression on technology

1. Leisure-income and labor-income models

   (a) The leisure-income model is a standard model of consumer behavior with preferences defined over consumption of leisure and spending on all other goods.

   The labor-income model is the equivalent model expressed in terms of net trades. It is the simplest model incorporating both factor supply and consumption and can be embedded into a model with production.

   (b) Define λ as consumption of leisure time and Y as consumption of goods. The individual has an endowment of $\bar{L}$ time (actually, leisure time) that he can sell to obtain consumer goods. We also suppose the individual has some non-endowment source of numeraire:

   $$\pi > 0$$

   The consumption set has the usual nonnegativity restriction plus the requirement that the consumer can not consume more leisure than he has in his endowment:

   $$\lambda \leq \bar{L}$$

   We suppose the wage rate is w and consumer goods are numeraire.

   (c) In the notation of the gross consumption model:

   $$\tilde{X} = \{(\lambda, Y) \in \Re_+^2 | \lambda \leq \bar{L}\}$$

   $$\tilde{x} = \begin{bmatrix} \lambda \\ Y \end{bmatrix} \quad \omega = \begin{bmatrix} \bar{L} \\ 0 \end{bmatrix} \quad q = \begin{bmatrix} w \\ 1 \end{bmatrix}$$

Recall the general formula for the budget set:

$$\tilde{B}(q,\pi) = \{\tilde{x} \in \tilde{X} \mid q\tilde{x} \leq q\omega + \pi\}$$

We have $q\tilde{x} = (w,1)'(\lambda, Y) = w\lambda + Y$ and $q\omega + \pi = (w,1)'(\bar{L}, 0) + \pi = w\bar{L} + \pi$. The budget set is then:

$$w\lambda + Y \leq w\bar{L} + \pi, \;\; 0 \leq \lambda \leq \bar{L}, \;\; Y \geq 0$$

Notice that if $\pi = 0$ then the constraint says that foregone earnings plus spending equals total possible earnings.

(d) Labor supply appears explicitly when we translate the model into net trades.

Labor supply (defined this way) is:

$$L = \bar{L} - \lambda \geq 0$$

Then:

$$x = \tilde{x} - \omega = \begin{bmatrix} \lambda \\ Y \end{bmatrix} - \begin{bmatrix} \bar{L} \\ 0 \end{bmatrix} = \begin{bmatrix} -L \\ Y \end{bmatrix}$$

Replacing $\tilde{x}$ with $x+\omega$ in each restriction on the consumption set, $\tilde{X}$, gives the net trades set, X:

$$-L + \bar{L} \geq 0, \;\; \text{so} \;\; -L \geq -\bar{L}$$

$$Y + 0 \geq 0, \;\; \text{so} \;\; Y \geq 0$$

$$-L + \bar{L} \leq \bar{L}, \;\; \text{so} -L \leq 0$$

The budget set also requires $qx \leq \pi$, so $w(-L) + Y \leq \pi$. The budget set is therefore:

$$w(-L) + Y \leq \pi, \;\; 0 \geq -L \geq -\bar{L}, \;\; Y \geq 0$$

Figure 1

2. Basic structure of the optimal tax model

(a) There is a single price-taking and utility-maximizing consumer.

This will be changed later to allow for many consumers.

Diamond and Mirrlees state:

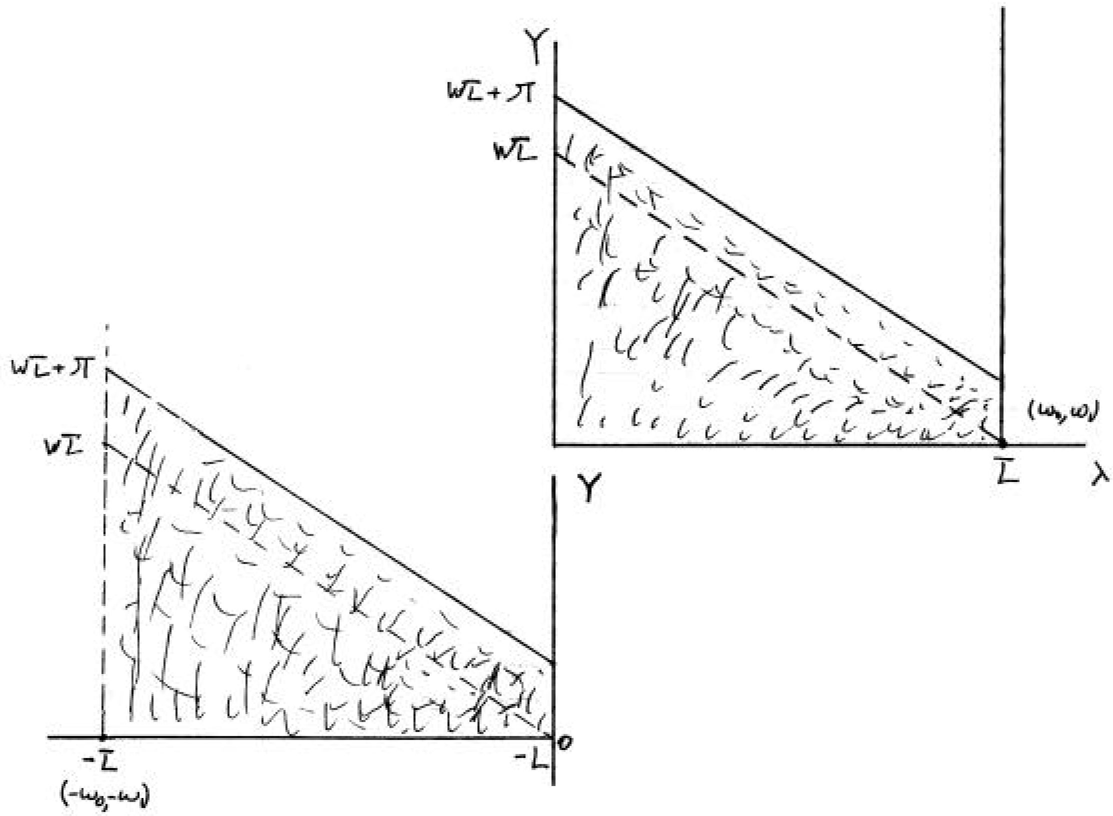

Figure 1

> The general situation we want to discuss is an economy in which there are many consumers, public and private production, public consumption, and many different kinds of feasible tax instruments. We think it is easier to understand the problem if we present the analysis first for a single consumer, no public consumption, and only commodity taxation, *although this case has no intrinsic interest* (emphasis added). The main point of the paper is that the analysis of this special case carries over in the main to the general analysis.

Even later on, however, their analysis of optimal commodity taxation (which is what we are mostly interested in) assumes no *private* production. This sets aside the need to tell a story about how the government interacts with private producers. I think this is a bit too stylized, so I vary it below.

(b) All production takes place within a single price-taking and profit-maximizing firm (see Mas-Colell, Chapter 5.E).

(c) There are $n+1$ *commodities* or *goods*, numbered "0" to n.

  i. Good "0" will always be "time."

  ii. As noted above, time is special for a number of reasons.

    A. It is not possible to consume more than one's endowment of it. In particular, one cannot liquidate other components of one's endowment vector in order to buy more than one's endowment of time.

    B. Many authors assume that labor is the only factor of production. The reasons for this are discussed later. In general we will *not* make this assumption.

  iii. Note that some inputs into production processes may also be consumed by individuals.

    The ith commodity may be strictly positive in an individual's gross consumption vector and strictly negative in a firm's netput vector.

    Again, the most important example of this is time.

(d) The government is going to make her choices first, then the consumer will make his.

Agents will take the government's choices as fixed when making their choices.

The government will anticipate the subsequent behavior of agents and the effects on the economy.

(e) We have already motivated the need to formulate the model in terms of consumer and producer prices.

The vector of *consumer prices* is denoted:

$$q \equiv (q_0, ..., q_n) \in \Re^{n+1}$$

The vector of *producer prices* is denoted:

$$p \equiv (p_0, ..., p_n) \in \Re^{n+1}$$

NOTE: This is the notation in most of this literature. Auerbach, however, uses the opposite notation!

3. Government

   (a) The government chooses a *public consumption vector*:

   $$x^G \equiv (x_0^G, ..., x_n^G) \in \Re_+^{n+1}$$

   (b) Think of x^G as a vector of commodities that the government will buy from firms and give to the consumer.

   (c) Since there is just one consumer we do not need to specify how x^G is divided up among people. We also do not need to distinguish private goods from public goods.

   (d) We will treat x^G as exogenous.

   Diamond and Mirrlees do this for most of their analysis. They then show that making x^G a choice variables does not change the form of the optimal tax formulas or the analysis of those formulas (1971, II; section IX).

   (e) The government either directly or indirectly chooses a vector of unit taxes:

   $$t \equiv (t_0, ..., t_n) \in \Re^{n+1}$$

4. Consumers

   (a) Primitive preferences are defined over gross consumption:

   $$\tilde{U}(\tilde{x} + x^G)$$

   To express this in terms of net trades, eliminate $\tilde{x}$ using $\tilde{x} = x + \omega$ and define the new function U:

   $$U(x) \equiv \tilde{U}(x + x^G + \omega)$$

   (the constants are suppressed).

(b) We have already discussed gross and net budget sets and taxation of net trades. We now want to include the possibility of an additional source of numeraire from profits earned by firms.

Denote these profits:

$$\pi(p)$$

Note! This depends on producer prices and brings these prices into the budget set, when it is nonzero. The budget set in net trades is then:

$$B(q,p) = \{x \in X | qx \leq \pi(p)\}$$

(c) Utility Maximization:

$$\begin{array}{cl} \underset{x}{\text{Max}} & U(x) \\ \text{s.t.} & qx = \pi(p) \end{array}$$

i. The Lagrangian is (note that we introduce α here):

$$\mathcal{L} = U(x) + \alpha[\pi(p) - qx]$$

ii. This gives the *vector* of demands:

$$x[q, \pi(p)]$$

Note that if $\pi(p) = 0$ then this is homogeneous of degree zero in q, the consumer prices. Recall the constraint above: if $\pi(p) = 0$ then multiplying q by a constant has no effect on the constraint.

iii. This gives indirect utility:

$$V[q, \pi(p)]$$

From the envelope theorem, α is the marginal utility of income, $\alpha \geq 0$, and:

$$\frac{\partial V}{\partial q_k} = -\alpha x_k$$

Note that this says that indirect utility decreases with a price increase, as we should expect.[1]

(d) Expenditure Minimization:

$$\begin{array}{cl} \underset{x}{\text{Min}} & qx \\ \text{s.t.} & U(x) = \bar{u} \end{array}$$

The Lagrangian is:

$$\mathcal{L} = qx + \hat{\alpha}[\bar{u} - U(x)]$$

[1] A little warning: α in Auerbach is defined to be the marginal *social* utility of income. We come to that later, just note that his α and this α are not the same. Our notation more closely follows Atkinson-Stiglitz and Myles.

This gives the *vector* of compensated demand:

$$x^c(q, \bar{u})$$

This gives the expenditure function:

$$E(q, \bar{u})$$

From the envelope theorem, $\hat{\alpha}$ is the reciprocal of the marginal utility of income, $\hat{\alpha} \geq 0$, and:

$$\frac{\partial E}{\partial q_k} = x_k^c(q, \bar{u})$$

This holds identically at the solution to the Lagrangian. The "Slutksy matrix" comes from taking the derivatives with the prices. Define the convenient "S" notation:

$$\begin{bmatrix} S_{00} & S_{01} & \cdots & S_{0,n} \\ S_{10} & S_{11} & \cdots & S_{1,n} \\ \vdots & \vdots & \vdots & \vdots \\ S_{n,0} & S_{n,1} & \cdots & S_{n,n} \end{bmatrix} \equiv \begin{bmatrix} \frac{\partial^2 E}{\partial q_0 \partial q_0} & \frac{\partial^2 E}{\partial q_0 \partial q_1} & \cdots & \frac{\partial^2 E}{\partial q_0 \partial q_n} \\ \frac{\partial^2 E}{\partial q_1 \partial q_0} & \frac{\partial^2 E}{\partial q_1 \partial q_1} & \cdots & \frac{\partial^2 E}{\partial q_1 \partial q_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 E}{\partial q_n \partial q_0} & \frac{\partial^2 E}{\partial q_n \partial q_1} & \cdots & \frac{\partial^2 E}{\partial q_n \partial q_n} \end{bmatrix}$$

Then:

$$\begin{bmatrix} S_{00} & S_{01} & \cdots & S_{0,n} \\ S_{10} & S_{11} & \cdots & S_{1,n} \\ \vdots & \vdots & \vdots & \vdots \\ S_{n,0} & S_{n,1} & \cdots & S_{n,n} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_0^c}{\partial q_0} & \frac{\partial x_0^c}{\partial q_1} & \cdots & \frac{\partial x_0^c}{\partial q_n} \\ \frac{\partial x_1^c}{\partial q_0} & \frac{\partial x_1^c}{\partial q_1} & \cdots & \frac{\partial x_1^c}{\partial q_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x_n^c}{\partial q_0} & \frac{\partial x_n^c}{\partial q_1} & \cdots & \frac{\partial x_n^c}{\partial q_n} \end{bmatrix}$$

5. Production

   (a) A *netput vector* is a list of commodities:

   $$y = (y_0, ..., y_n)$$

   with inputs into production processes measured negatively and outputs measured positively.

   (b) We assume that all production takes place within a single price-taking and profit-maximizing firm (see Mas-Colell, Chapter 5.E).

   Note that while we are assuming that one firm produces all of the commodities, we are not assuming that only one commodity is produced. The one firm controls the production process for each commodity.

(c) We assume that there are no produced inputs into production, so there are only primary factors of production. This is the assumption of *no intermediate goods*.

Without this assumption we would, at the very least, have to allow for tax wedges in transactions between production processes (analogous to transactions between firms). This would necessitate many sets of producer prices. There is a small literature on this subject.

Later on, we will assume there is no simultaneity in the production of outputs: production of one output is functionally separate from the production of any other output. This is the assumption of *no joint production*.

I do not think this assumption is really needed except when we want to make producer prices independent of demand (see below).

It is also true that these two assumptions together imply that the technology for producing each good can be described by the standard kind of production function.

(d) We represent aggregate technology using the transformation function. This can be written in implicit form:

$$F(y) = 0$$

(e) Profit Maximization:

$$\underset{y}{\text{Max}} \quad py$$
$$\text{s.t.} \quad F(y) = 0$$

i. Goods supply and factor demand:

$$y(p)$$

Note that this is homogeneous of degree zero in producer prices. Multiplying all producer prices by a constant is a positive transformation of the objective function and will not change the optimizing choices.

ii. Aggregate profits function:

$$\pi(p) \equiv py(p)$$

If we assume CRS then there are zero profits, so:

$$py(p) = 0$$

(f) Notice that the maximization problem will give $n + 1$ conditions:

$$p_i = \lambda \frac{\partial F}{\partial y_i}, i = 0, ..., n$$

If we take the ratio with the first equation then we have the n conditions:

$$\frac{p_i}{p_0} = \frac{\partial F / \partial y_i}{\partial F / \partial y_0}, i = 1, ..., n$$

If we can further suppose (as we do later) that $p_0 = 1$, then:

$$p_i = \frac{\partial F/\partial y_i}{\partial F/\partial y_0}, i = 1, ..., n$$

Now, without any serious loss of generality we can suppose:

$$\partial F(y)/\partial y_0 = 1, \ \ \forall y$$

This holds as long as the technology can be written in the standard way:

$$F(y) \equiv y_0 - f(y_1, ..., y_n)$$

In this case $\frac{\partial F}{\partial y_0} = 1$ at all netput vectors.
The conclusion is:

$$p_i = \partial F/\partial y_i, i = 1, ..., n$$

The vector of produce prices equals the gradient of the technology. This is a convenient fact used over and over again in this literature, often without explanation or with contradictory explanations.

6. A digression on technology

I noted above that the transformation function and the discussion of technology bury quite a lot.

The technical discussion below is motivated by the discussion in Kemp et al. (1978) and material in Chapter 3 of Arrow and Hahn ("General Competitive Analysis"). Feel free to skip it – or read it and improve it!

(a) The production set for an individual firm incorporates one constraint, *technological possibility.*

The production set for the entire economy will also have only this constraint if it is just the aggregation of production sets for each firm. This is how it is defined in Mas-Collel, Chapter 5.E.

(b) This not the only way to define the aggregate production set. It may absorb an additional constraint, from the aggregate resources available.

Technology and the aggregate resource constraint define *feasibility.* This is clearly what Kemp et al. (1978) have in mind and seems to be the only way that derivatives of the "transformation surface" will correspond to the slope of the production possibilities frontier.

(c) The following is the transformation function, as defined in Kemp et al. (1978), under the assumptions of no intermediate goods and no joint production.

Commodities 0 through p are primary factors. Commodities $p+1$ through n are then produced goods 1 through $n-p$, respectively.

For each of the produced goods, production possibilities are described by a production function:

$$y_{p+i} = f_i(-y_{0i}, ..., -y_{pi}), \ i = 1, ..., n-p$$

where $(y_{0i}, ..., y_{pi})$ is the amount of each primary factor used to produce y_{p+i}.

The production possibilities set for the economy, or the set of feasible netput vectors, is:

$$\begin{aligned} Y \ = \ & \Big\{ (y_0, ..., y_n) \in \Re^{n+1} \mid \exists \, (y_{0i}, ..., y_{pi}) \geq 0, \ i = 1, ..., n-p \\ & \text{such that } y_{p+i} \leq f_i(-y_{0i}, ..., -y_{pi}), \ i = 1, ..., n-p, \\ & \text{and } \sum_{i=1}^{n-p} (-y_{0i}, ..., -y_{pi}) = (-y_0, ..., -y_p) \leq (\omega_0, ..., \omega_p) \Big\} \end{aligned}$$

The transformation function, F, is the upper boundary of Y.

(d) To describe this in more detail takes a little work.

The basic idea is that each factor total (specified by $y_0, ..., y_p$) must satisfy the respective aggregate resource constraint (specified by $\omega_0, ..., \omega_p$), not necessarily with equality though, and factors are allocated across production processes so that output y_{p+i} equals $f_i(.)$ for all $i = 1, ..., n-p$, *and* it is not possible to increase all outputs.

i. Recall that the last condition does not hold simply because y_{p+i} equals $f_i(.)$ for all $i = 1, ..., n-p$. Efficient production requires inputs to be correctly allocated to each production process, and not merely for each output to be at the maximum for arbitrary inputs.

If we maximize y_n subject to the constraints above we obtain a relationship of the form:

$$y_n = \tilde{F}(y_0, ..., y_{n-1})$$

The transformation function is then just:

$$F(y_0, ..., y_n) \equiv \tilde{F}(y_0, ..., y_{n-1}) - y_n$$

The production possibilities set for the economy can then be written:

$$Y = \Big\{ (y_0, ..., y_n) \in \Re^{n+1} \mid F(y) \leq 0 \Big\}$$

(e) Marginal rate of transformation

Given a point $\bar{y}$ such that $F(\bar{y}) = 0$ and two outputs l and k, the marginal rate of transformation of l for k is:

$$\frac{\partial F / \partial y_l}{\partial F / \partial y_k}$$

This is just the negative of the slope of the production possibilities frontier (with l on the horizontal axis and k on the vertical axis) as discussed in intermediate micro. Note that total inputs are fixed at the levels specified by $\bar{y}$.