**Lecture 17**

## Migration Equilibrium

1. Utility Hills

   (a) The first piece of machinery we need for understanding migration equilibrium is "utility hills."[1]

   For this, we return to the individual allocations that are feasible under equal treatment.

   We then trace out the *maximum* utility an individual could achieve as more people are added.

   Figure 1 presents the "hilly" case.

   **Figure 1**

   Other outcomes are certainly possible.

   If preferences are tilted towards $X^i$, then it is possible that the overall ("variable number of region") optimum has one-person communities and no public good. The utility "hill" would be monotone decreasing in population.

   If preferences are tilted towards $G$, then it is possible that the overall optimum has a single community and no private good. The utility "hill" would be monotone increasing in population.

   **Figure 2**

   (b) Formally, for $N_j > 0$, the "utility hill" for individual $i$ in region $j$ is the function $V_j^i(N_j)$:

   $$V_j^i(N_j) = \max_{X_j^i, G_j} U(G_j, X_j^i) \quad \text{subject to:} \quad N_j X_j^i + G_j = f_j(N_j)$$

   For the case $N_j = 0$, it is common to define:

   $$V_j^i(0) = \lim_{N_j \to 0} V_j^i(N_j)$$

   provided the limit exists.[2]

[1] The term comes from Marcus Berliant, who wants to know more about when they exist and whether they look like a hill.

[2] Note that a limit may be infinite and still exist.

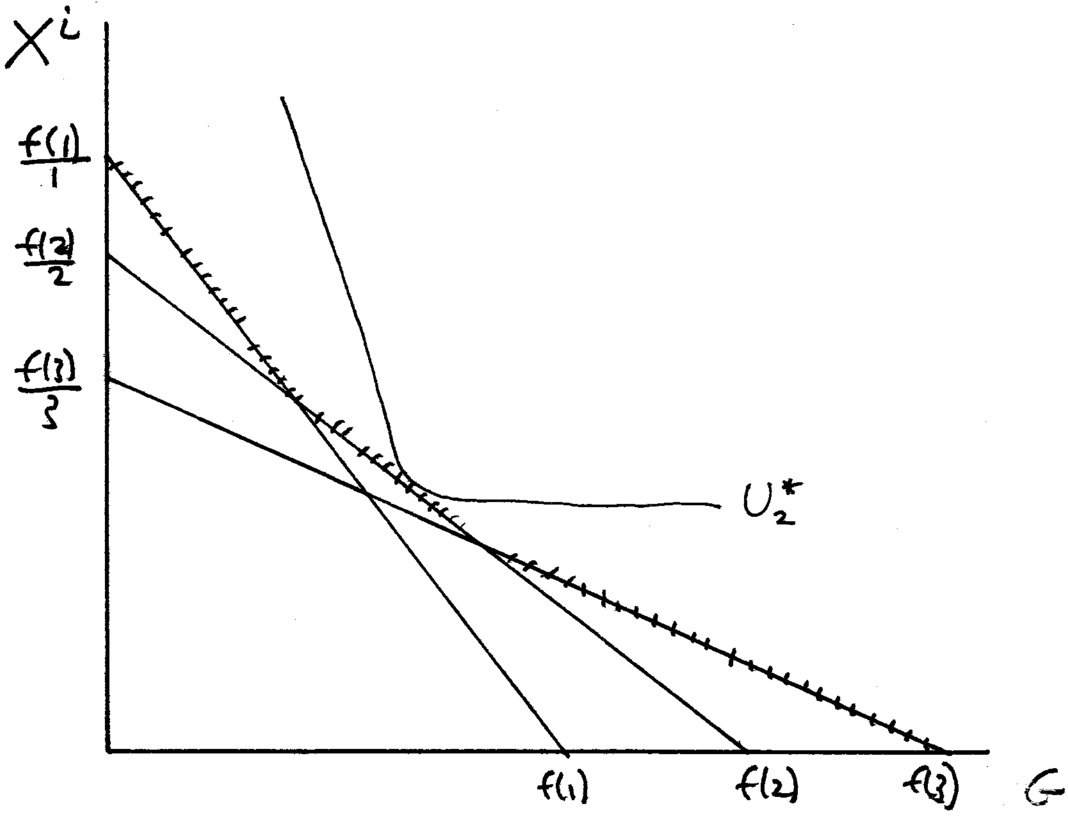
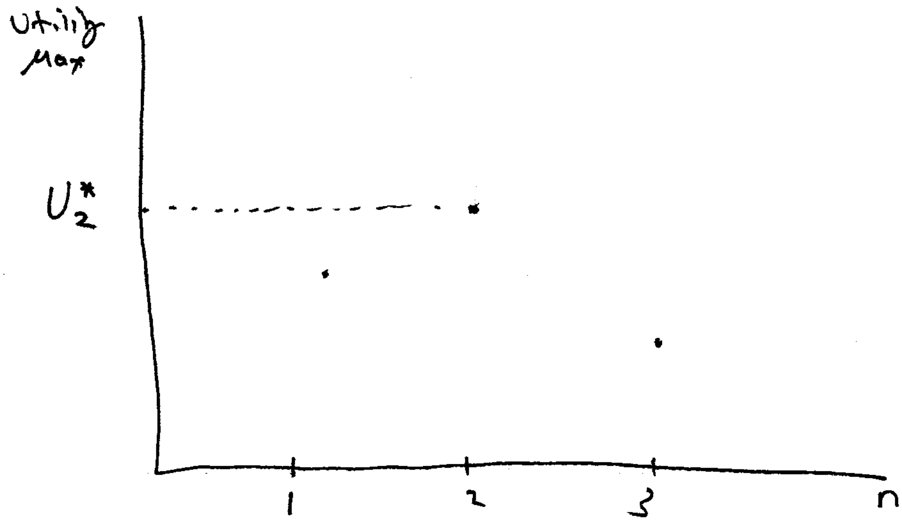
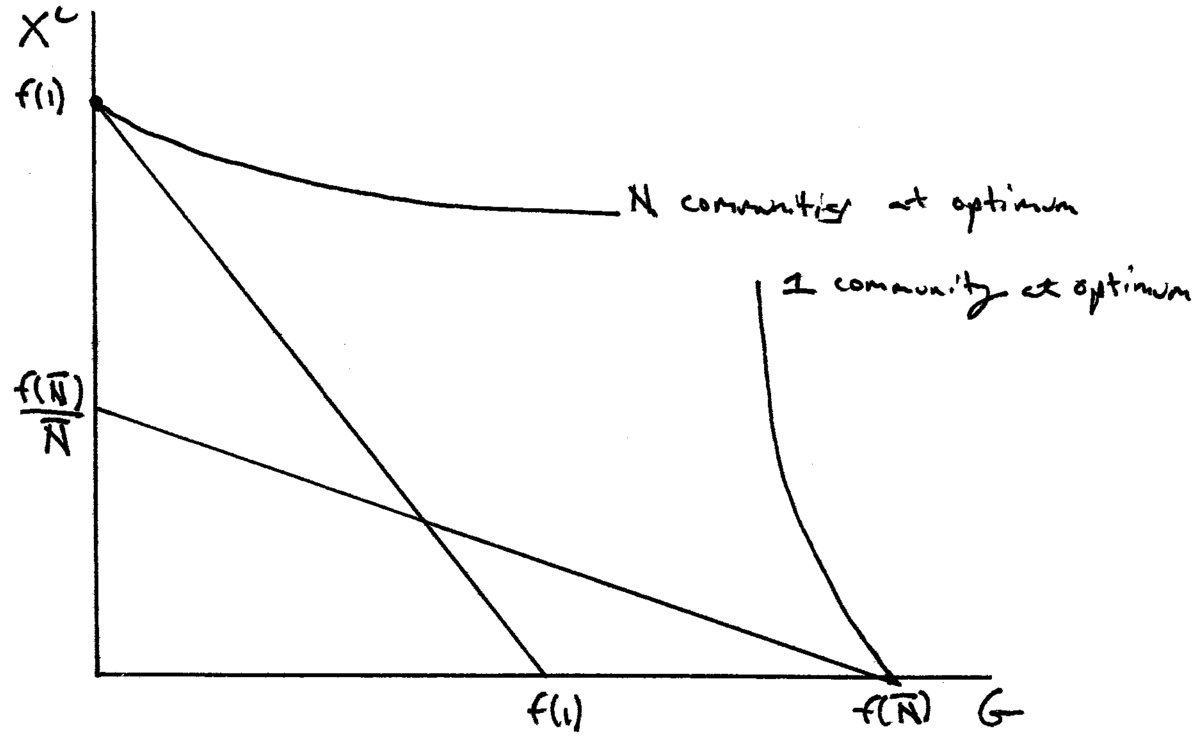



Figure 1





$X^i$
f(1)
$\frac{f(\bar{N})}{\bar{N}}$
N communities at optimum
1 community at optimum
f(1)
$f(\bar{N})$
G


Figure 2

(c) By definition, if $N_j > 0$, the utility hill tells us the maximum utility person $i$ in region $j$ obtains when the population is $N_j$.[3]

(d) Our construction assumes all agents are small and identical. Suppose we also assume that immigrants and original occupants must be treated the same.

It now follows that $V_j^i$ gives the maximum utility an immigrant into region $j$ could obtain.

(e) If $N_j = 0$ then the interpretation is trickier. In a related context, Atkinson-Stiglitz write:

> [A]n individual must form a conjecture about what his utility would be if there is no one of exactly his type within the community. For instance, if there are not doctors within a community, a doctor would have to conjecture the wages that a doctor would be paid (after tax). We assume that these conjectures are correct. (p. 541)

Note, however, that this problem is strictly an artifact of the assumption that individuals are small. If individuals are large then it never arises: an individual would examine $V_j^i(1)$, the utility he would obtain after becoming the sole (large) resident of region $j$.

2. Definition of migration equilibrium

*Notation shift.*
The analysis of multi-community models requires many subscripts. It is superfluous now to keep using $i$ for an individual. We therefore drop this, and use $i$ to index communities.

Assume we have $J$ possible communities ("jurisdictions"). Note that the communities need not be identical – they could have different amounts of the fixed factor, for example. A *migration equilibrium* is a vector $(n_1, ..., n_J)$ such that $n_i \geq 0$ for all $i$, $\sum_{i=1}^{J} n_i = \bar{N}$, and

$$n_i > 0 \Rightarrow V_i(n_i) \geq V_j(n_j), \text{ all } i, j$$

[3] We say "obtains" and not "could obtain": it is assumed that the resource is properly allocated between private and public good.

3. Existence of migration equilibria.

   The usual reference is Ginsburgh, V., Papageorgiou, Y.Y., Thisse, J.-F., 1985, *On existence and stability of spatial equilibria and steady-states*, Regional Science and Urban Economics 15, 149158. They use a fixed point argument.

   Other approaches are possible. Rothstein (2006) uses the theorem of the maximum to show the existence of an equilibrium (in a mobile capital model) and establish the continuity of the allocation in the characteristics.[4]

4. Characterization of migration equilibria.

   (a) Theorem 1.
   Suppose we have a vector $n = (n_1, ..., n_J)$ such that $n_i \geq 0$ for all $i$ and $\sum_{i=1}^{J} n_i = N$.
   Then $n$ is a migration equilibrium if and only if all regions $i, j$ with $n_i > 0$ and $n_j > 0$ satisfy:

   $$V_i(n_i) = V_j(n_j)$$

   and all regions $i, k$ with $n_i > 0$ and $n_k = 0$ satisfy:

   $$V_i(n_i) \geq V_k(n_k)$$

   Proof.
   Suppose $n$ is a migration equilibrium. Consider the case $n_i > 0$ and $n_j > 0$. By definition of migration equilibrium we have $V_i(n_i) \geq V_j(n_j)$ and $V_j(n_j) \geq V_i(n_i)$. Therefore $V_i(n_i) = V_j(n_j)$ as required. Now consider the case $n_i > 0$ and $n_k = 0$. We have $n_i > 0$, so by definition of migration equilibrium we have $V_i(n_i) \geq V_k(n_k)$ as required.

   Now suppose we have the equality and inequality conditions. We want to show that $n$ is a migration equilibrium. Fix $n_i > 0$ and any region $j$. If $n_j > 0$ then the equality condition gives $V_i(n_i) = V_j(n_j)$. Therefore $V_i(n_i) \geq V_j(n_j)$. If instead $n_j = 0$ then the inequality condition gives $V_i(n_i) \geq V_j(n_j)$. Thus the latter holds for all $j$, so we have a migration equilibrium.

   (b) Theorem 2.
   Suppose we have a vector $n = (n_1, ..., n_J)$ such that $n_i > 0$ for all $i$ and $\sum_{i=1}^{J} n_i = N$.
   Then $n$ is a migration equilibrium if and only if all regions $i, j$ satisfy:

   $$V_i(n_i) = V_j(n_j)$$

[4] *Discountinuous payoffs, shared resources, and games of fiscal competition: Existence of pure strategy Nash Equilibrium*, Journal of Public Economic Theory, forthcoming.

Proof.
An immediate corollary of the previous result.

5. Finding migration equilibria.

   In general, finding all of the possible migration equilibria is a brute force exercise. You have to consider all of the following cases:

   (a) Each region has all of the population. Verify that no individual wants to migrate.
   (b) Each pair of regions has all of the population. Verify that the proposed allocation of the population is feasible, it implies a common level of utility in both regions, and verify that no individual wants to migrate.
   (c) Each triple of regions is occupied. Etc.
   (d) Etc.

   If you know in advance that *all* regions must be occupied in equilibrium, however, then all you have to do is verify that the proposed allocation of the population is feasible, it implies a common level of utility in all regions, and verify that no individual wants to migrate. A sufficient condition to ensure that all regions are occupied in equilibrium is:

$$\lim_{n_i \to 0} V_i(n_i) = +\infty$$

   We return to this below.

   The figure considers the case of two regions.

**Figure 3**

6. Stability of migration equilibrium.

   A migration equilibrium is (locally) stable if a small deviation from the equilibrium population causes utility in the region that gains population to be a little lower than utility in the region that loses population.

   (a) If this occurs then the deviation will not feed on itself. People in the region that lost population will not want to follow those who left.
   (b) This property certainly holds if utility in the region gaining population falls and utility in the region losing population rises.
       Note that this is a statement about absolute utility changes. It is *stronger* than the definition, which is a statement about relative utility changes.

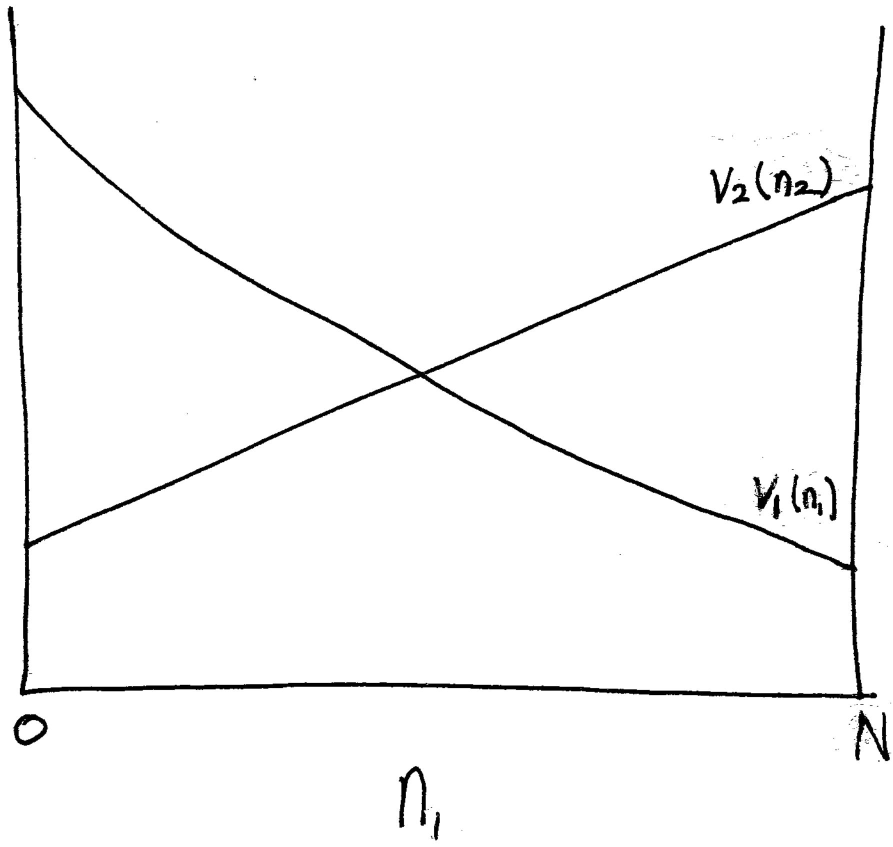



Figure 3

<u>Note:</u> $V_2(n_2)$ is actually $V_2(N-n_1)$. It is conventional to write the simpler expression.

7. Stability and comparative statics of migration equilibrium

   Stability rules out counter-intuitive comparative statics.

   Suppose there is a technology shock in region 1 making it more productive. Or, more interestingly, suppose region 1 is systematically providing incorrect levels of the local public good and then improves itself.

   **Figure 4**

   Setting aside the question of how society would move to the new equilibrium, the implication of instability is clear: the region that improves itself must have *less* population in the new equilibrium. This is completely unintuitive!

   We usually assume stability and use whatever convenient properties it implies. We sometimes even assume something stronger than stability and justify the assumption on the basis that it gives us stability.[5]

8. Formal comparative statics of migration equilibrium.

   If we know every region is occupied in every equilibrium, then the following conditions must hold:

   $$V_2(n_2) = V_1(n_1)$$

   $$V_3(n_3) = V_1(n_1)$$

   $$V_J(n_J) = V_1(n_1)$$

   $$n_1 + ... + n_J = \bar{N}$$

   Assuming differentiability, we can use the implicit function theorem to derive *migration equilibrium functions*:

   $$n_1(.), n_2(.), ...n_J(.)$$

   where each function depends on the characteristics of all regions.

   We can differentiate these functions with respect to the characteristics and derive comparative statics for the equilibrium populations.

   This is only a local result, however, if the number of regions occupied in equilibrium changes as we vary the characteristics. This is quite likely – although the situation isn't entirely chaotic. We return to this point later in the course.

[5] Yes, that is a little woolly. I'm just reporting the facts.

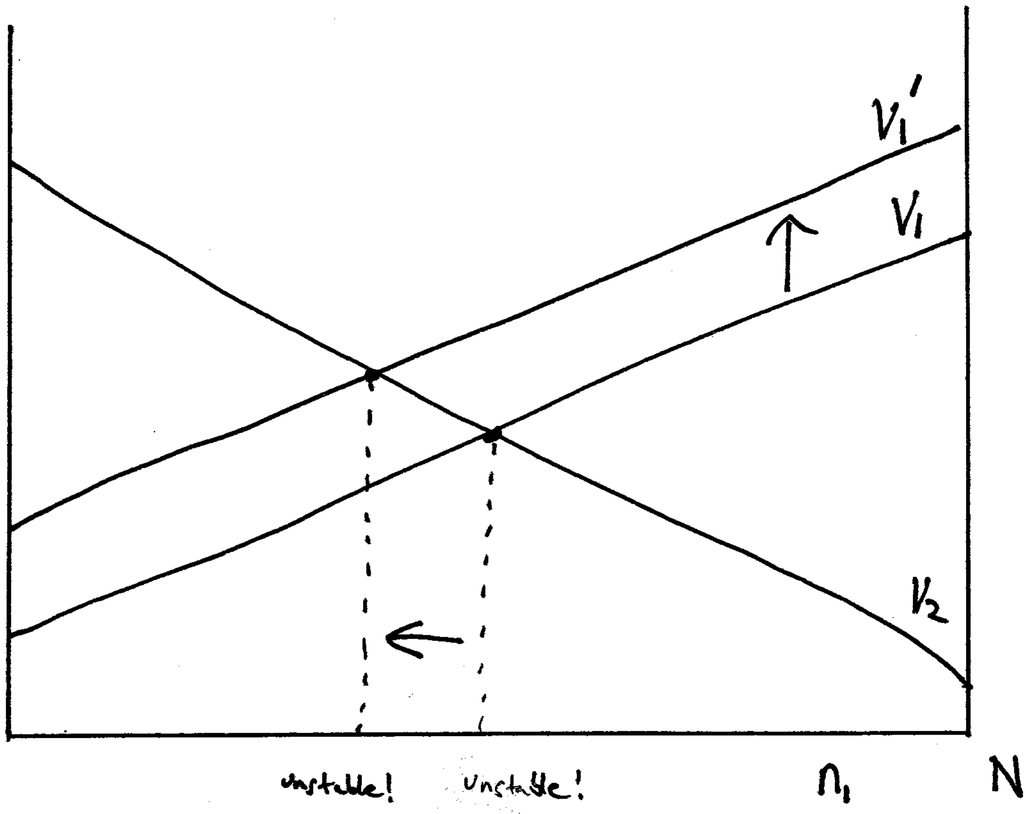



Figure 4

Region 1 becomes more attractive but its equilibrium population falls!

Unstable equilibria have weird comparative statics.

9. A trivial result.

   Theorem.
   Suppose there are exactly two regions. Both regions are identical, each utility hill has one peak and is symmetric around the peak, and you can assign everyone to the communities so they achieve the peak utility level. Then this allocation of the population is a stable migration equilibrium.

   Under these assumptions the utility hills (drawn as functions of $N_1$) coincide. Thus, every split of the population is a migration equilibrium. Symmetry ensures that utility in one region after it gains population is equal to the utility in the other region after it loses population, so the equilibria are stable.

   If there are more than two regions, we need to think about each utility hill on its own (as a function of $N_j$, say). In this case an equal split of the population gives everyone the common peak level of utility, so this is a migration equilibrium. Is it stable?

10. All that can go wrong.

    Each utility hill has one peak, but there are two exogenously specified communities that are not identical.

    The equilibrium that provides the highest common level of utility is unstable.

**Figure 5**

11. Overall:

    (a) Existence does not seem to be much of a problem as long as all of the basic underlying functions are continuous.

    (b) Uniqueness seems virtually guaranteed not to hold.

    (c) Stability is quite problematic.

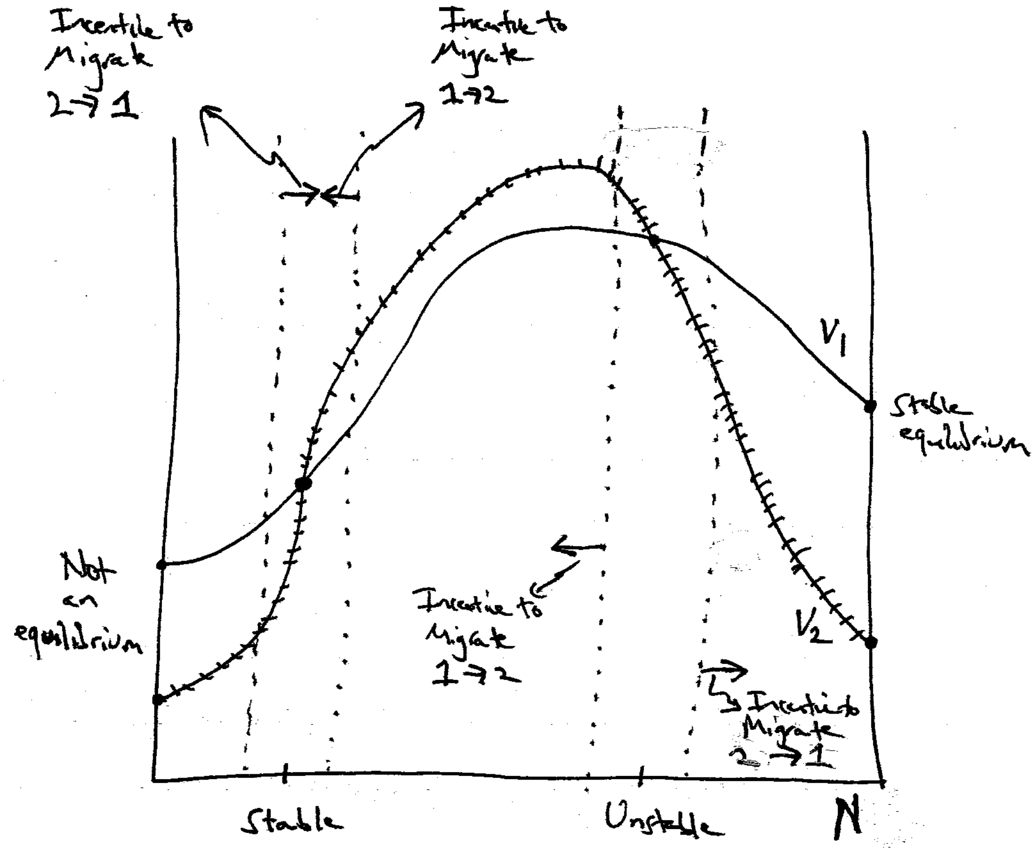

Incentive to Migrate 2→1
Incentive to Migrate 1→2
V1
Stable equilibrium
Not an equilibrium
Incentive to Migrate 1→2
V2
Incentive to Migrate 2→1
Stable
Unstable
N

Figure 5