Lecture 16

Introduction

1. We now begin a more detailed examination of:
   (a) Properties of efficient allocations in models with regions.
   (b) Efficiency of equilibrium allocations in models with regions.

   We mostly (but not entirely) set aside question of existence per se.

2. In evaluating whether an allocation is efficient, the fundamental conditions that must be met are:
   (a) whether the number of communities is correct
   (b) whether the allocation of individuals among communities is correct, and
   (c) whether the level of expenditures on various public goods within the community is correct

3. Whether these conditions can be met in equilibrium depends critically on:
   (a) the objectives of local governments
   (b) the tax instruments available to them
   (c) the pattern of factor ownership
   (d) the way communities compete (large numbers or small numbers)

   We will come back to these points time and again.

Pareto Problems

1. Feasible Equal Treatment Allocations

   (a) Allocations that give all agents in the same community the same allocation play a large role in the formal analysis of multi-community models.

   One reason is that, if all agents have the same preferences and endowments, then these are the only allocations that could be sustained in equilibrium.

   Another reason is that, even if people have different preferences, as long as they have the same endowments then these are the only allocations that could be sustained in equilibrium if everyone shares the costs of public goods equally.

   Of course, the restriction to feasible allocations that provide equal treatment makes the analysis easier at every juncture.

   (b) An important necessary condition for the optimal population size in a community emerges from the analysis of feasible equal treatment allocations. In other words, we learn something important about optimal populations even before we introduce specific preferences.

   We examine this in two closely related frameworks.

2. Exogenous productive resource model.

   The simplest framework for analyzing multi-community models assumes each individual is endowed with one unit of "all purpose good."

   One unit of all-purpose good can be transformed into one unit of private good or one unit of public good.

   Preferences for individual i are defined over private good and public good:

   $$U(G, X^i)$$

   We use X^i instead of X to distinguish between per-capita private good and aggregate private good.

   The question now is, as people are added to the community, how does the set of feasible equal-treatment allocations change? If there are N people in the community, the aggregate resource constraint is clearly:

   $$NX^i + G = NY^i$$

   Furthermore, the set of feasible equal-treatment allocations is fully characterized by the set of feasible equal-treatment *individual* allocations. These are:

   $$X^i + \frac{G}{N} = Y^i$$

As N varies, we have:

Figure 1

So, what is the optimal number of people to have in one community? In this case the answer is, “all of them.”

3. Endogenous productive resource model: The Henry George Theorem.

   Now suppose people are endowed, not with all purpose good, but with a unit of labor that they “inelastically” supply wherever they reside. All purpose good is produced from labor, and as before one unit of it can be transformed into one unit of private good or one unit of public good.

   The standard assumptions on this technology are:

   $$f(0) = 0, f' > 0, f'' < 0$$

   Note the contrast with Tiebout.

   (a) He emphasizes differences in demand of individuals (could be due to differences in tastes or incomes or both).
   (b) He also un-couples where people work and where they consume.

   Again, as people are added to the community, how does the set of feasible equal-treatment allocations change? If there are N people in the community, the aggregate resource constraint is now:

   $$NX^i + G = f(N)$$

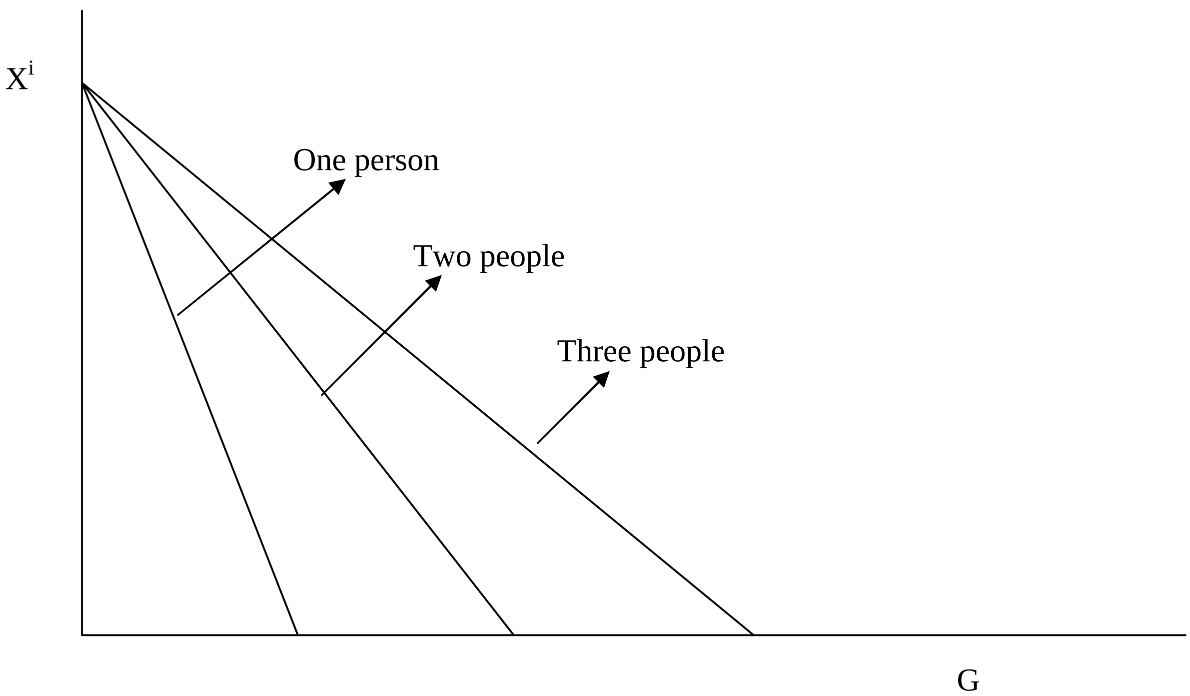
   The set of feasible equal-treatment *individual* allocations is:

   $$X^i + \frac{G}{N} = \frac{f(N)}{N}$$

   Given the assumptions on f, the term $f(N)/N$ decreases as N increases. So, now there is a cost to adding people to the community: there is less per-capita productive resource. There is still the benefit of everyone sharing the non-congestible local public good, though. This creates a tradeoff.

Figure 1

Individual Feasible Equal-Treatment Allocations with an Exogenous Productive Resource



Visually, we have:

Figure 2

So, what is the optimal number of people to have in one community? To answer this, we fix G arbitrarily and choose N to maximize per-capita private good:

$$\max_{N} \quad \frac{f(N) - G}{N}$$

The first order condition is:

$$\frac{Nf'(N) - f(N) + G}{N^2} = 0$$

Thus:

$$G = f(N) - Nf'(N) \tag{1}$$

This has two interpretations, both of which are important:

(a) Suppose worker/consumers are paid the marginal product of labor. Then this says that at the optimal population, the return to fixed factors ("land rent") exactly covers the cost of the local public good.

This is known as the Henry George Theorem.

(b) Alternatively, since $f(N) = G + NX^i$, we have $f(N) - NX^i = f(N) - Nf'(N)$, so:

$$f'(N) = X^i \tag{2}$$

This says that individuals should be added to the community until their marginal product just equals their allocation of private good.

When this holds the individual just "carries his or her weight," creating no surplus or deficit.

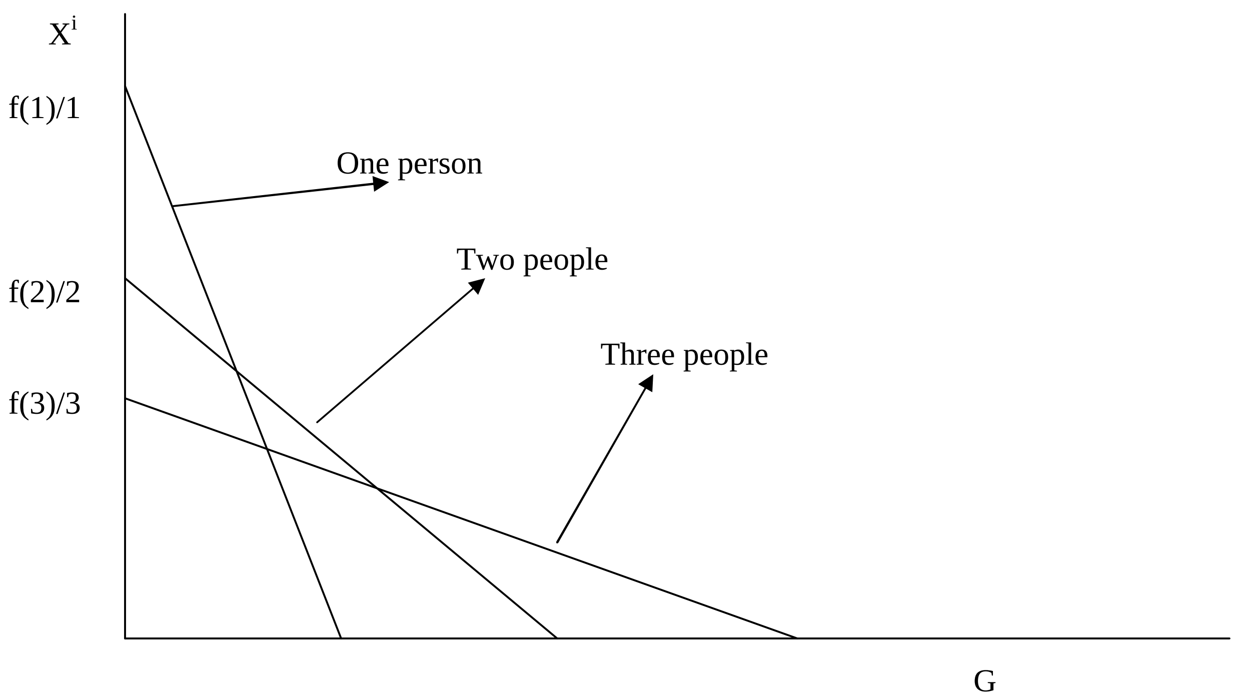
It is *very interesting* and *not always recognized* that these conditions emerge from the analysis of feasibility alone. Preferences play no role in these conclusions!

Note, finally, that the set of feasible equal-treatment individual allocations is *not convex*. Indeed, the frontier is convex to the origin.

This has some implications below.

Figure 2

Individual Feasible Equal-Treatment Allocations with an Endogenous Productive Resource



4. Atkinson-Stiglitz (17-1, 17-2) derive the conditions for an efficient quantity of public good and an efficient population in a single community. We now consider this analysis.

   Note that if we set aside the integer problem, we can replicate the community and absorb the entire population. We therefore compute the efficient number of communities as well.

5. Pareto Problem #1 (one-type and an endogenous number of communities).[1]

   The model is the same as in the endogenous productive resource model.

   Note that the production possibilities frontier is:

   $$f(N) = NX^i + G = X + G$$

   where X is total production of private good.

   Note that $MRT_{GX} = 1$. With N given we have:

   $$F(G, X) = f(N) - X - G$$

   Thus:

   $$MRT_{GX} = \frac{\partial F/\partial G}{\partial F/\partial X} = 1$$

   (a) The social planner solves:

   $$\max_{X^i, G, N} U(G, X^i) \quad \text{subject to:} \quad NX^i + G = f(N)$$

   (b) Lagrangian:

   $$\mathcal{L} = U(G, X^i) + \lambda[f(N) - NX^i - G]$$

   (c) First order conditions:

   $$U_X - \lambda N = 0$$

   $$U_G - \lambda = 0$$

   $$\lambda[f'(N) - X^i] = 0$$

[1] We will always assume there is no integer problem.

(d) The first two conditions give:

$$N\frac{U_G}{U_X} = 1$$

This is the Samuelson condition. The left hand side is the sum of MRS_{GX} at the optimum, since all individuals are identical and consume the same bundle. The right hand side equals MRT_{GX}.

(e) The third condition gives:

$$f'(N) = X^i$$

We know from the previous analysis that this condition must hold. It is reassuring that it comes back explicitly!

Figure 3

(f) The optimal number of communities is then $\bar{N}/N^*$, where $\bar{N}$ is the total number of individuals in society and N^* is the solution to the problem above.

6. Variation on #1.

(a) Write the technology more generally, to allow for congestion:

$$f(N) = X + C(G, N)$$

(b) The social planner now solves:

$$\begin{array}{ll} \underset{X^i, G, N}{\text{Max}} \; U(G, X^i) & \\ \text{subject to:} & NX^i + C(G, N) = f(N) \end{array}$$

(c) Lagrangian:

$$\mathcal{L} = U(G, X^i) + \lambda[f(N) - NX^i - C(G, N)]$$

(d) First order conditions:

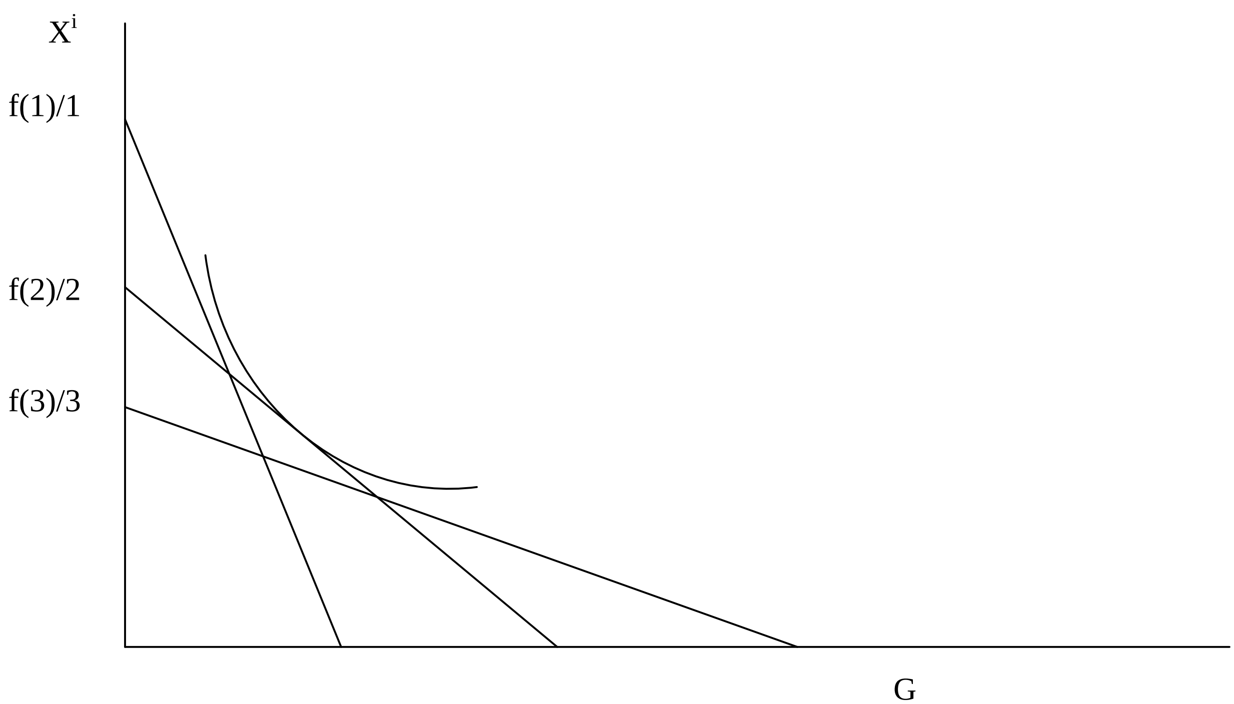
$$U_X - \lambda N = 0$$

$$U_G - \lambda C_G = 0$$

$$\lambda[f'(N) - X^i - C_N] = 0$$

Figure 3

Optimal Community with Two People
The Samuelson condition and the location efficiency condition
will hold



(e) The first two conditions give:

$$N\frac{U_G}{U_X} = C_G$$

The exact meaning of this is a little tricky. It purports to be completely general. It still holds if G is a publicly provided private good. In that case, however, the right hand side must be $(N)\text{MRT}_{GX}$.

This is what is going on. The MRT_{GX} tells us how much of good X must be foregone for society to have one extra unit of G. The C_G tells us how much of good X must be foregone for society to provide each individual with an extra unit of G. If G is a publicly provided private good, then society must forego N times as many units of X as are needed to create one extra unit of the good.

The bottom line is that the relationship between C_G and MRT_{GX} depends on the nature of G. Furthermore, while it is conventional to call this condition the Samuelson condition, this isn't quite right.

(f) The Henry George Theorem does not hold, at least not in the simplest form. We now have:

$$f'(N) = X^i + C_N$$

Therefore:

$$Nf'(N) = NX^i + NC_N = f(N) - C(G, N) + NC_N$$

Therefore at the optimum:

$$C(G, N) = f(N) - Nf'(N) + NC_N$$

The hypothetical return to fixed factors ("land rent") would no longer cover the cost of the local public good. Additional resources are needed, equal to the revenue that would be raised from taxing every individual at the marginal congestion cost.

7. Pareto Problem #2 (one-type and two communities).

We now move to what *may* be a more "real-world" framework. We assume that the number of communities is exogenously fixed. Thus, we can no longer simply create communities of optimal size, enough to absorb the given population. Now there are a fixed number of communities and the entire population must be allocated between them.

(a) Model.

i. There are 2 jurisdictions:

$$j = 1, 2$$

ii. Preferences are still identical for all individuals, but in order to have an unambiguous notation for the derivatives we will write:

$$U_j(G_j, X_j^i)$$

It should be understood that

$$(G_1, X_1^i) = (G_2, X_2^i) \Longrightarrow U_1(G_1, X_1^i) = U_2(G_2, X_2^i)$$

iii. Production in jurisdiction j:

$$f_j(N_j)$$

Production possibilities frontier:

$$f_j(N_j) = X_j + C_j(G_j, N_j)$$

iv. Total population condition:

$$N_1 + N_2 = \bar{N}$$

(b) Optimization

i. Choose G_j, X_j^i, and N_j for $j = 1, 2$ to maximize $U(G_1, X_1^i)$ subject to the constraints:

Fixed utility in region 2:

$$U_2(G_2, X_2^i) = \bar{U}_2$$

Over all communities, the total production of all-purpose good equals all uses:

$$\sum_j f_j(N_j) - \sum_j N_j X_j^i - \sum_j C_j(G_j, N_j) = 0$$

All households must reside somewhere:

$$N_1 + N_2 = \bar{N}$$

ii. Lagrangian:

$$\begin{aligned}
\mathcal{L} &= U_1(G_1, X_1^i) \\
&+ \lambda[U_2(G_2, X_2^i) - \bar{U}_2] \\
&+ \mu[f_1(N_1) + f_2(N_2) - N_1 X_1^i - N_2 X_2^i - C_1(G_1, N_1) - C_2(G_2, N_2)] \\
&+ \psi[\bar{N} - N_1 - N_2]
\end{aligned}$$

For convenience, define:

$$F_j(N_j) = f_j'(N_j)$$

First order conditions with $X_1^i, X_2^i, G_1, G_2, N_1, N_2$ respectively give:

$$U_{12} - \mu N_1 = 0$$

$$\lambda U_{22} - \mu N_2 = 0$$

$$U_{11} - \mu C_{11} = 0$$

$$\lambda U_{21} - \mu C_{21} = 0$$

$$\mu(F_1 - X_1^i - C_{12}) - \psi = 0$$

$$\mu(F_2 - X_2^i - C_{22}) - \psi = 0$$

(c) Results.

The first four conditions give the Samuelson condition in both communities:

$$N_j \frac{U_{j1}}{U_{j2}} = C_{j1}, \quad j = 1, 2$$

The last two conditions give "locational efficiency":

$$F_1 - X_1^i - C_{12} = F_2 - X_2^i - C_{22}$$

If we assume no congestion and linear transformation between X_j^i and G_j then:

$$C_j(G_j, N_j) = G_j$$

Then the locational efficiency condition would be:

$$F_1 - X_1^i = F_2 - X_2^i$$

This says that the marginal net contribution of an additional individual must be the same across communities.

Further discussion of these optimality conditions is in Flatters et al. (1974) and Myers (1990). The derivation above closely follows Myers (1990).

(d) How does this locational efficiency condition compare to the one in the first Pareto problem? Does the Henry George Theorem hold here? These questions are closely related. We consider them in turn.

Since we are investigating a "pure" Henry George Theorem, we assume $C_j(G_j, N_j) = G_j$.

Suppose we repeat the two-community optimization, *but without the total population constraint*. Then:

$$\begin{aligned} \mathcal{L} &= U(G_1, X_1^i) \\ &+ \lambda[U(G_2, X_2^i) - \bar{U}_2] \\ &+ \mu[f_1(N_1) + f_2(N_2) - N_1 X_1^i - N_2 X_2^i - C_1(G_1, N_1) - C_2(G_2, N_2)] \end{aligned}$$

The first order conditions with N_1 and N_2 give:

$$\mu(F_1 - X_1^i) = 0$$

$$\mu(F_2 - X_2^i) = 0$$

This gives:

$$F_1 - X_1^i = F_2 - X_2^i = 0$$

Thus, the optimal population condition from the first Pareto problem would hold in both communities.

We should therefore expect, in general, that $F_j - X_j^i \neq 0$. The reason is that the aggregate population in Problem #2, $\bar{N}$, is not in general equal to the optimal aggregate population. This would be $N_1^* + N_2^*$ such that $F_j - X_j^i = 0$ for $j = 1, 2$.

(e) When (and only when) we have $F_1 - X_1^i = F_2 - X_2^i = 0$ do we have an "aggregate Henry George theorem".

The proof proceeds along the same lines as the earlier one.

For more on this, see Hartwick (1980).